

SEGAL-BARGMANN TRANSFORM AND PALEY-WIENER THEOREMS ON HEISENBERG MOTION GROUPS

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ABSTRACT. We study the Segal-Bargmann transform on the Heisenberg motion groups $\mathbb{H}^n \ltimes K$, where \mathbb{H}^n is the Heisenberg group and K is a compact subgroup of $U(n)$ such that (K, \mathbb{H}^n) is a Gelfand pair. The Poisson integrals associated to the Laplacian for the Heisenberg motion group are also characterized using Gutzmer's formulae. Explicitly realizing certain unitary irreducible representations of $\mathbb{H}^n \ltimes K$, we prove the Plancherel theorem. A Paley-Wiener type theorem is proved using complexified representations.

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1. INTRODUCTION

For any $f \in L^2(\mathbb{R}^n)$, it is easy to see that $f * \rho_t$ extends as an entire function to the whole of \mathbb{C}^n , where $\rho_t(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$ is the heat kernel on \mathbb{R}^n and the image of $L^2(\mathbb{R}^n)$ under the map $f \rightarrow f * \rho_t$ can be characterized as the Hilbert space of entire functions on \mathbb{C}^n which are square integrable with respect to the positive weight $\rho_{t/2}(y) dx dy$.

The mapping $f \rightarrow f * \rho_t$ is called the Segal-Bargmann transform, also known as the coherent state transform or the heat kernel transform. Segal and Bargmann independently proved in the 1960's in the context of quantum mechanics that this transform is a unitary map from $L^2(\mathbb{R}^n)$ onto $\mathcal{O}(\mathbb{C}^n) \cap L^2(\mathbb{C}^n, \mu)$, where $d\mu(z) = \frac{1}{(2\pi t)^{n/2}} e^{-\frac{|y|^2}{2t}} dx dy$ and $\mathcal{O}(\mathbb{C}^n)$ denotes the space of entire functions on \mathbb{C}^n .

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This transform has attracted a lot of attention in the recent years mainly due to the work of Hall [4] where a similar result was established for an arbitrary compact connected Lie group K . He introduced a generalization of the Segal-Bargmann transform on a compact connected Lie group. If K is such a group, this coherent state transform maps $L^2(K)$ isometrically onto the space of holomorphic functions in $L^2(G, \mu_t)$, where G is the complexification of K and μ_t is an appropriate positive heat kernel measure on G . The generalized coherent state transform is defined in terms of the heat kernel on the compact group K and its analytic continuation to the complex group G . Similar results have been proved for compact symmetric spaces by Stenzel in [12].

For the Heisenberg group \mathbb{H}^n , Krötz et al proved in [7] that the image of $L^2(\mathbb{H}^n)$ under the heat kernel transform is not a weighted Bergman space with a non-negative weight, but can be considered as a direct integral of twisted Bergman spaces. Similar results for non compact symmetric spaces have been proved in [5] and [6].

Next, consider the following result on \mathbb{R} due to Paley and Wiener. A function $f \in L^2(\mathbb{R})$ admits a holomorphic extension to the strip $\{x + iy : |y| < t\}$ such that

$$\sup_{|y| \leq s} \int_{\mathbb{R}} |f(x + iy)|^2 dx < \infty \quad \forall s < t$$

if and only if

$$(1.1) \quad \int_{\mathbb{R}} e^{s|\xi|} |\tilde{f}(\xi)|^2 d\xi < \infty \quad \forall s < t$$

where \tilde{f} denotes the Fourier transform of f .

The condition (1.1) is the same as

$$\int_{\mathbb{R}} |\widetilde{e^{s\Delta^{\frac{1}{2}}} f(\xi)}|^2 d\xi < \infty \quad \forall s < t$$

where Δ is the Laplacian on \mathbb{R} . This point of view was explored by R. Goodman in Theorem 2.1 of [2].

The condition (1.1) also equals

$$\int_{\mathbb{R}} |e^{i(x+iy)\xi}|^2 |\tilde{f}(\xi)|^2 d\xi < \infty \quad \forall |y| < t.$$

Here $\xi \mapsto e^{i(x+iy)\xi}$ may be seen as the complexification of the parameters of the unitary irreducible representations $\xi \mapsto e^{ix\xi}$ of \mathbb{R} . The above point of view was further developed by R. Goodman in [3] (see Theorem 3.1). Similar results were established for the Euclidean motion group $M(2)$ of the plane \mathbb{R}^2 in [9] and in the context of general motion groups $\mathbb{R}^n \ltimes K$, where K is a compact subgroup of $SO(n)$ in [11]. Aim of this paper is to prove results analogous to the above three, for the Heisenberg motion groups $\mathbb{HM} = \mathbb{H}^n \ltimes K$, where \mathbb{H}^n is the Heisenberg group and K is a compact subgroup of $U(n)$ such that (K, \mathbb{H}^n) is a Gelfand pair.

The plan of this paper is as follows : In the following section the range of the Segal-Bargmann transform on \mathbb{HM} is characterized as a direct integral of weighted Bergman spaces. The third section is devoted to a study of Poisson integrals on \mathbb{HM} by using a Gutzmer formula for compact Lie groups established by Lassalle in 1978 (see [8]) and a Gutzmer formula on \mathbb{C}^{2n} . This is modelled after the work of Goodman [2]. In the final section we prove the Plancherel theorem on \mathbb{HM} thereby listing the unitary, irreducible representations on which Plancherel measure rests. Then we prove a Paley-Wiener type theorem which characterizes functions extending holomorphically to the complexification of \mathbb{HM} which is an analogue of Theorem 3.1 of [3].

2. SEGAL-BARGMANN TRANSFORM

In this section we want to study the Segal-Bargmann transform on the Heisenberg motion group. We recall that, for the Heisenberg group \mathbb{H}^n , it was proved by Krötz et al in [7] that the image of $L^2(\mathbb{H}^n)$ under the heat kernel transform is not a weighted Bergman space with a non-negative weight, but can be considered as a direct integral of twisted Bergman spaces. Here we prove that a similar result is true for Heisenberg motion groups as well.

2.1. Segal-Bargmann transform for the special Hermite semigroup.

Let $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ be the Heisenberg group with the group operation defined by

$$(z, t) \cdot (w, s) = (z + w, t + s + \frac{1}{2} \operatorname{Im}(z \cdot \overline{w})) \quad \text{where } z, w \in \mathbb{C}^n, t, s \in \mathbb{R}.$$

Alternatively, we can consider \mathbb{H}^n as $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with the group law

$$(x, y, t)(u, v, s) = (x + u, y + v, t + s + \frac{1}{2}(u \cdot y - v \cdot x)) \quad \text{for } x, y, u, v \in \mathbb{R}^n, t, s \in \mathbb{R}.$$

The center of \mathbb{H}^n is $\{(0, 0, t) : t \in \mathbb{R}\}$.

Let $\lambda \in \mathbb{R}$, $\lambda \neq 0$. For suitable functions f on \mathbb{H}^n , let us define a function f^λ on \mathbb{R}^{2n} by

$$f^\lambda(x, u) = \int_{\mathbb{R}} f(x, u, t) e^{i\lambda t} dt.$$

For $f, g \in L^1(\mathbb{R}^{2n})$, the λ -twisted convolution of f and g is defined by

$$f *_\lambda g(x, u) = \int_{\mathbb{R}^{2n}} f(x', u') g(x - x', u - u') e^{-i\frac{\lambda}{2}(u \cdot x' - x \cdot u')} dx' du'.$$

Then, we have for Schwartz functions $f, g \in \mathcal{S}(\mathbb{H}^n) = \mathcal{S}(\mathbb{R}^{2n+1})$,

$$(f * g)^\lambda = f^\lambda *_\lambda g^\lambda$$

where $*$ denotes the group convolution on \mathbb{H}^n .

Let \mathcal{L} denote the sublaplacian on the Heisenberg group defined by

$$\mathcal{L} = - \sum_{j=1}^n (X_j^2 + Y_j^2)$$

where $X_j, Y_j, j = 1, 2, \dots, n$ together with $T = \frac{\partial}{\partial t}$ forms a basis for the Heisenberg Lie algebra. For the explicit expressions for these vector fields we refer to [15]. The heat kernel for \mathcal{L} is denoted by p_t and its inverse Fourier transform in the central variable is explicitly given by (see [7])

$$p_t^\lambda(x, u) = (4\pi)^{-n} \left(\frac{\lambda}{\sinh \lambda t} \right)^n e^{-\frac{\lambda}{4} \coth(\lambda t)(|x|^2 + |u|^2)}.$$

It can be shown that p_t^λ is the heat kernel associated to the special Hermite operator L_λ . That is, $e^{-tL_\lambda}f = f *_\lambda p_t^\lambda$, for $f \in L^2(\mathbb{R}^{2n})$. Define a positive weight function \mathcal{W}_t^λ on \mathbb{C}^{2n} by

$$\mathcal{W}_t^\lambda(x + iy, u + iv) = 4^n e^{\lambda(u \cdot y - v \cdot x)} p_{2t}^\lambda(2y, 2v) \quad \text{where } x, y, u, v \in \mathbb{R}^n.$$

Let $\mathcal{B}_t^\lambda(\mathbb{C}^{2n})$ be the weighted Bergman space defined as

$$\mathcal{B}_t^\lambda(\mathbb{C}^{2n}) = \{F \text{ entire on } \mathbb{C}^{2n} : \int_{\mathbb{C}^{2n}} |F(z, w)|^2 \mathcal{W}_t^\lambda(z, w) dz dw < \infty\}.$$

Theorem 2.1. *The map $e^{-tL_\lambda} : f \rightarrow f * p_t^\lambda$, called the λ -twisted heat kernel transform, is a unitary operator from $L^2(\mathbb{R}^{2n})$ to $\mathcal{B}_t^\lambda(\mathbb{C}^{2n})$.*

For more details and the proof see [7].

2.2. Laplacian and heat kernel on Heisenberg motion groups.

Let K be a compact, connected Lie subgroup of $\text{Aut}(\mathbb{H}^n)$, such that (K, \mathbb{H}^n) is a Gelfand pair. By this we mean that the convolution algebra of K -invariant L^1 -functions on \mathbb{H}^n is commutative. A maximal compact connected group of automorphisms of \mathbb{H}^n is given by the unitary group $U(n)$ acting on \mathbb{H}^n via $k(z, t) = (kz, t)$. Conjugating by an automorphism of \mathbb{H}^n if necessary, we can always assume that $K \subset U(n)$. It is well known that $(U(n), \mathbb{H}^n)$ is a Gelfand pair and there are many proper subgroups K of $U(n)$ for which (K, \mathbb{H}^n) form a Gelfand pair.

Let \mathbb{HM} be the semidirect product of \mathbb{H}^n and K with the group law

$$(x, y, t, k)(u, v, s, h) = ((x, u, t) \cdot (k \cdot (u, v), s), kh) \quad \text{where } (x, y, t), (u, v, s) \in \mathbb{H}^n; k, h \in K.$$

\mathbb{HM} is called the Heisenberg motion group. For $K = U(n)$, $\mathbb{HM}^n = \mathbb{H}^n \ltimes U(n)$ is more commonly known as the Heisenberg motion group. However, in this paper by a Heisenberg motion group \mathbb{HM} , we shall mean $\mathbb{H}^n \ltimes K$. Points in \mathbb{HM} will be denoted by (x, y, t, k) where $(x, y, t) \in \mathbb{H}^n$ and $k \in K$.

Let K_1, K_2, \dots, K_N be an orthonormal basis of the Lie algebra \underline{k} of K . In the Heisenberg motion group \mathbb{HM} , we have $2n + 1 + N$ one parameter subgroups given

by

$$\begin{aligned}
G_j &= \{(te_j, 0, 0, I) : t \in \mathbb{R}\} \\
G_{n+j} &= \{(0, te_j, 0, I) : t \in \mathbb{R}\} \\
G_{2n+1} &= \{(0, 0, t, I) : t \in \mathbb{R}\} \\
G_{2n+1+l} &= \{(0, 0, 0, \exp(tK_l)) : t \in \mathbb{R}\}
\end{aligned}$$

where $1 \leq j \leq n$, $1 \leq l \leq N$ and e_j are the co-ordinate vectors in \mathbb{R}^n . Corresponding to these one parameter subgroups we have $2n + 1 + N$ left invariant vector fields $X_1, X_2, \dots, X_{2n+1+N}$, which form a basis of the Lie algebra of \mathbb{HM} . The Laplacian Δ on \mathbb{HM} is given by

$$\Delta = -(X_1^2 + X_2^2 + \dots + X_{2n+1+N}^2).$$

Let $Sp(n, \mathbb{R})$ denote the symplectic group consisting of order $2n$ real matrices with determinant one that preserve the symplectic form $[(x, u), (y, v)] = u \cdot y - v \cdot x$. Let $O(2n, \mathbb{R})$ be the orthogonal group of order $2n$. Define $M = Sp(n, \mathbb{R}) \cap O(2n, \mathbb{R})$. Then there is a one to one correspondence between M and the unitary group $U(n)$. Let $k = a + ib$ be an $n \times n$ complex matrix with real and imaginary parts a and b . Then k is unitary if and only if the matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in M$. A simple computation using the above and the fact that $K \subset U(n)$ shows that

$$\Delta = -\Delta_{\mathbb{H}^n} - \Delta_K$$

where $\Delta_{\mathbb{H}^n} = \sum_{j=1}^{2n+1} X_j^2$ and Δ_K are the Laplacians on \mathbb{H}^n and K respectively.

Since $\Delta_{\mathbb{H}^n}$ and Δ_K commute, it follows that the heat kernel ψ_t associated to Δ is given by the product of the heat kernels k_t on \mathbb{H}^n and q_t on K . In other words

$$\begin{aligned}\psi_t(x, u, \xi, k) &= k_t(x, u, \xi)q_t(k) \\ &= \left((4\pi)^{-n} \int_{\mathbb{R}} e^{-i\lambda\xi} e^{-t\lambda^2} p_t^\lambda(x, u) d\lambda \right) \left(\sum_{\pi \in \hat{K}} d_\pi e^{-\frac{\lambda_\pi t}{2}} \chi_\pi(k) \right).\end{aligned}$$

Here, for each unitary, irreducible representation π of K , d_π is the degree of π , λ_π is such that $\pi(\Delta_K) = -\lambda_\pi I$ and $\chi_\pi(k) = \text{tr}(\pi(k))$ is the character of π . For more details, see [4].

2.3. Segal-Bargmann transform.

Denote by G the complexification of K . Let κ_t be the fundamental solution at the identity of the following equation on G :

$$\frac{du}{dt} = \frac{1}{4} \Delta_G u,$$

where Δ_G is the Laplacian on G , for details please see [4]. It should be noted that κ_t is the real, positive heat kernel on G which is not the same as the analytic continuation of q_t on K .

Define $\mathcal{A}_t^\lambda(\mathbb{C}^{2n} \times G)$ to be the weighted Bergman space

$$\mathcal{A}_t^\lambda(\mathbb{C}^{2n} \times G) = \{F \text{ entire on } \mathbb{C}^{2n} \times G : \int_G \int_{\mathbb{C}^{2n}} |F(z, w, g)|^2 \mathcal{W}_t^\lambda(z, w) dz dw d\nu(g) < \infty\}$$

where

$$d\nu(g) = \int_K \kappa_t(xg) dx \text{ on } G.$$

We now introduce a measurable structure on $\bigsqcup_{\lambda \neq 0} \mathcal{A}_t^\lambda(\mathbb{C}^{2n} \times G)$. By a section s of

$\bigsqcup_{\lambda \neq 0} \mathcal{A}_t^\lambda(\mathbb{C}^{2n} \times G)$ we mean an assignment

$$\begin{aligned}s : \mathbb{R}^* &\rightarrow \bigsqcup_{\lambda \neq 0} \mathcal{A}_t^\lambda(\mathbb{C}^{2n} \times G) \\ \lambda &\rightarrow s_\lambda \in \mathcal{A}_t^\lambda(\mathbb{C}^{2n} \times G).\end{aligned}$$

Now we define a direct integral of Hilbert spaces by

$$\int_{\mathbb{R}^*}^{\oplus} \mathcal{A}_t^\lambda(\mathbb{C}^{2n} \times G) e^{2t\lambda^2} d\lambda = \left\{ s : \mathbb{R}^* \rightarrow \bigsqcup_{\lambda \neq 0} \mathcal{A}_t^\lambda(\mathbb{C}^{2n} \times G) \text{ such that } s \text{ is measurable and} \right. \\ \left. \|s\|^2 = \int_{\mathbb{R}^*} \|s_\lambda\|_\lambda^2 e^{2t\lambda^2} d\lambda < \infty \right\}$$

where $\|\cdot\|_\lambda$ denotes the norm in $\mathcal{A}_t^\lambda(\mathbb{C}^{2n} \times G)$. Clearly this is a Hilbert space.

For suitable functions f on $\mathbb{H}\mathbb{M}$, let us define a function f^λ on $\mathbb{R}^{2n} \times K$ by

$$f^\lambda(x, u, k) = \int_{\mathbb{R}} f(x, u, t, k) e^{i\lambda t} dt.$$

Notice that this definition is consistent with the one used earlier for functions on \mathbb{H}^n (i.e. for right K -invariant functions on $\mathbb{H}\mathbb{M}$). Then we have the following theorem :

Theorem 2.2. *If $f \in L^2(\mathbb{H}\mathbb{M})$, then $f * \psi_t$ extends holomorphically to $\mathbb{C}^{2n+1} \times G$.*

- (a) *The image of $L^2(\mathbb{H}\mathbb{M})$ under the Segal-Bargmann transform $f \rightarrow f * \psi_t$ cannot be characterized as a weighted Bergman space with a non-negative weight.*
- (b) *For every $t > 0$, the Segal-Bargmann transform $e^{-t\Delta} : L^2(\mathbb{H}\mathbb{M}) \rightarrow \bigsqcup_{\lambda \neq 0} \mathcal{A}_t^\lambda(\mathbb{C}^{2n} \times G)$, $f \rightarrow (f * \psi_t)^\lambda$ is an isometric isomorphism.*

Proof. Let $f \in L^2(\mathbb{H}\mathbb{M})$. Expanding f in the K -variable using the Peter-Weyl theorem we obtain

$$f(x, u, t, k) = \sum_{\pi \in \widehat{K}} d_\pi \sum_{i,j=1}^{d_\pi} f_{ij}^\pi(x, u, t) \phi_{ij}^\pi(k)$$

where for each $\pi \in \widehat{K}$, d_π is the degree of π , ϕ_{ij}^π 's are the matrix coefficients of π and $f_{ij}^\pi(x, u, t) = \int_K f(x, u, t, k) \overline{\phi_{ij}^\pi(k)} dk$. Here, the convergence is understood in the L^2 -sense. Moreover, by the universal property of the complexification of a compact Lie group (see Section 3 of [4]), all the representations of K , and hence all the matrix entries, extend holomorphically to G .

Since ψ_t is K -invariant (as a function on \mathbb{H}^n) a simple computation shows that

$$f * \psi_t(x, u, t, k) = \sum_{\pi \in \widehat{K}} d_\pi e^{-\frac{\lambda_\pi t}{2}} \sum_{i,j=1}^{d_\pi} f_{ij}^\pi * k_t(x, u, t) \phi_{ij}^\pi(k),$$

where the convolution on the right is the one on \mathbb{H}^n . It is easily seen that $f_{ij}^\pi \in L^2(\mathbb{H}^n)$ for every $\pi \in \widehat{K}$ and $1 \leq i, j \leq d_\pi$. Hence $f_{ij}^\pi * k_t$ extends to a holomorphic function on \mathbb{C}^{2n+1} . We formally define the analytic continuation of $f * \psi_t$ to $\mathbb{C}^{2n+1} \times G$ by

$$f * \psi_t(z, w, \zeta, g) = \sum_{\pi \in \widehat{K}} d_\pi e^{-\frac{\lambda_\pi t}{2}} \sum_{i,j=1}^{d_\pi} f_{ij}^\pi * p_t(z, w, \zeta) \phi_{ij}^\pi(g)$$

where $(z, w, \zeta) \in \mathbb{C}^{2n+1}$ and $g \in G$.

We claim that the above series converges uniformly on compact subsets of $\mathbb{C}^{2n+1} \times G$ so that $f * \psi_t$ extends to an entire function on $\mathbb{C}^{2n+1} \times G$. We know from Section 4, Proposition 1 of [4] that the holomorphic extension of the heat kernel q_t on K is given by

$$q_t(g) = \sum_{\pi \in \widehat{K}} d_\pi e^{-\frac{\lambda_\pi t}{2}} \chi_\pi(g).$$

For each $g \in G$, define the function $q_t^g(k) = q_t(gk)$. Then q_t^g is a smooth function on K and is given by

$$\begin{aligned} q_t^g(k) &= \sum_{\pi \in \widehat{K}} d_\pi e^{-\frac{\lambda_\pi t}{2}} \chi_\pi(gk) \\ &= \sum_{\pi \in \widehat{K}} d_\pi e^{-\frac{\lambda_\pi t}{2}} \sum_{i,j=1}^{d_\pi} \phi_{ij}^\pi(g) \phi_{ji}^\pi(k). \end{aligned}$$

Since q_t^g is a smooth function on K , we have for each $g \in G$,

$$(2.1) \quad \int_K |q_t^g(k)|^2 dk = \sum_{\pi \in \widehat{K}} d_\pi e^{-\lambda_\pi t} \sum_{i,j=1}^{d_\pi} |\phi_{ij}^\pi(g)|^2 < \infty.$$

Let L be a compact set in $\mathbb{C}^{2n+1} \times G$. For $(z, w, \zeta, g) \in L$ we have,

$$(2.2) \quad |f * \psi_t(z, w, \zeta, g)| \leq \sum_{\pi \in \widehat{K}} d_\pi e^{-\frac{\lambda_\pi t}{2}} \sum_{i,j=1}^{d_\pi} |f_{ij}^\pi * p_t(z, w, \zeta)| |\phi_{ij}^\pi(g)|.$$

It is known that the inclusion $e^{-t\Delta_{\mathbb{H}^n}} L^2(\mathbb{H}^n) \hookrightarrow \mathcal{O}(\mathbb{C}^{2n+1})$ is continuous (see Section 3 of [7]) i.e. there exists a constant C_L depending on L such that for any $h \in L^2(\mathbb{H}^n)$,

$$\sup_{(z,w,\zeta) \in L'} |h * k_t(z, w, \zeta)| \leq C_L \|h\|_{L^2(\mathbb{H}^n)}$$

where L' is the projection of L to \mathbb{C}^{2n+1} . Using the above in (2.2) and applying Cauchy-Schwarz inequality we get

$$\begin{aligned} |f * \psi_t(z, w, \zeta, g)| &\leq C_L \sum_{\pi \in \widehat{K}} d_\pi \sum_{i,j=1}^{d_\pi} \|f_{ij}^\pi\|_{L^2(\mathbb{H}^n)} e^{-\frac{\lambda_\pi t}{2}} |\phi_{ij}^\pi(g)| \\ &\leq C_L \left(\sum_{\pi \in \widehat{K}} d_\pi \sum_{i,j=1}^{d_\pi} \int_{\mathbb{H}^n} |f_{ij}^\pi(X)|^2 dX \right)^{\frac{1}{2}} \left(\sum_{\pi \in \widehat{K}} d_\pi \sum_{i,j=1}^{d_\pi} e^{-\lambda_\pi t} |\phi_{ij}^\pi(g)|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Noting that $\|f\|_2^2 = \sum_{\pi \in \widehat{K}} d_\pi \sum_{i,j=1}^{d_\pi} \int_{\mathbb{H}^n} |f_{ij}^\pi(X)|^2 dX$ and q_t is a smooth function on G we prove the claim using (2.1). Hence $f * \psi_t$ extends holomorphically to $\mathbb{C}^{2n+1} \times G$.

Next, we want to prove the non-existence of a non-negative weight function W_t on $\mathbb{C}^{2n+1} \times G$ such that for every $f \in L^2(\mathbb{H}\mathbb{M})$, we have

$$\int_K \int_{\mathbb{H}^n} |f(x, u, t, k)|^2 dx du dt dk = \int_{\mathbb{C}^{2n+1} \times G} |f * \psi_t(z, w, \zeta, g)|^2 W_t(z, w, \zeta, g) dz dw d\zeta dg.$$

If we take f such that $f(x, u, t, k) = h(x, u, t)$ for $h \in L^2(\mathbb{H}^n)$, then we get from the above relation that

$$\int_{\mathbb{H}^n} |h(x, u, t)|^2 dx du dt = \int_{\mathbb{C}^{2n+1}} |h * k_t(z, w, \zeta)|^2 \left(\int_G W_t(z, w, \zeta, g) dg \right) dz dw d\zeta.$$

Since W_t is assumed to be non-negative, this clearly gives a contradiction to the fact that the image of $L^2(\mathbb{H}^n)$ under the Segal-Bargmann transform $h \rightarrow h * k_t$ cannot be characterized as a weighted Bergman space with a non-negative weight (see [7] for details).

For any $f \in L^2(\mathbb{H}\mathbb{M})$, define $f_X(k) = f(X, k)$ for $X \in \mathbb{H}^n$ and $k \in K$. Then using Theorem 2 in [4] we have

$$\begin{aligned} \|f\|_2^2 &= \int_{\mathbb{H}^n} \int_K |f(X, k)|^2 dk dX \\ &= \int_{\mathbb{H}^n} \int_G |f_X * q_t(g)|^2 d\nu(g) dX. \end{aligned}$$

Define $F_g(X) = f_X * q_t(g)$ for $g \in G$. Then, using Theorem 5.1 of [7] we get that

$$\begin{aligned} \|f\|_2^2 &= \int_G \int_{\mathbb{H}^n} |F_g(X)|^2 dX d\nu(g) \\ &= \int_G \int_{\mathbb{R}^*} e^{2t\lambda^2} \|(F_g * k_t)^\lambda\|_{\mathcal{B}_t^\lambda(\mathbb{C}^{2n})}^2 d\lambda d\nu(g) \\ &= \int_G \int_{\mathbb{R}^*} e^{2t\lambda^2} \int_{\mathbb{C}^{2n}} |(F_g * k_t)^\lambda(z, w)|^2 \mathcal{W}_t^\lambda(z, w) dz dw d\lambda d\nu(g). \end{aligned}$$

It is easy to see that $F_g * k_t(x, u) = f * \psi_t(x, u, g)$. Since the functions on both the sides extend holomorphically to \mathbb{C}^{2n} , we have $F_g * k_t(z, w) = f * \psi_t(z, w, g)$ for every $z, w \in \mathbb{C}^n$ and $g \in G$. Hence it follows that

$$\|f\|_2^2 = \int_{\mathbb{R}^*} \int_G e^{2t\lambda^2} \int_{\mathbb{C}^{2n}} |(f * \psi_t)^\lambda(z, w, g)|^2 \mathcal{W}_t^\lambda(z, w) dz dw d\nu(g) d\lambda.$$

It remains to prove the surjectivity part. Let $s \in \int_{\mathbb{R}^*}^\oplus \mathcal{A}_t^\lambda(\mathbb{C}^{2n} \times G) e^{2t\lambda^2} d\lambda$. Then $s_\lambda \in \mathcal{A}_t^\lambda(\mathbb{C}^{2n} \times G)$ and $\int_{\mathbb{R}^*} \|s_\lambda\|_\lambda^2 e^{2t\lambda^2} d\lambda < \infty$. Now $e^{-t\Delta_K} e^{-tL_\lambda}$ is a unitary map from $L^2(\mathbb{R}^{2n} \times K)$ onto $\mathcal{A}_t^\lambda(\mathbb{C}^{2n} \times G)$. So there exists $g_\lambda \in L^2(\mathbb{R}^{2n} \times K)$ for each $\lambda \neq 0$ such that $e^{-t\Delta_K} e^{-tL_\lambda} g_\lambda = e^{t\lambda^2} s_\lambda$ and $\int_{\mathbb{R}^*} \|g_\lambda\|_{L^2(\mathbb{R}^{2n} \times K)}^2 d\lambda < \infty$. This implies that there exists a unique $f \in L^2(\mathbb{HM})$ such that $f^\lambda(x, u) = g_\lambda(x, u)$ almost everywhere. Finally we have $(f * \psi_t)^\lambda = e^{-t\lambda^2} e^{-t\Delta_K} e^{-tL_\lambda} g_\lambda$ in L^2 . Using the above equalities we have $(f * \psi_t)^\lambda = s_\lambda$. This proves the surjectivity and hence the theorem. \square

3. GUTZMER'S FORMULA AND POISSON INTEGRALS

In this section, we briefly recall Gutzmer's formula on compact, connected Lie groups given by Lassalle in [8]. Then we prove a Gutzmer type formula for functions on \mathbb{C}^{2n} with respect to the K -action. With the help of the above Gutzmer's formulae, we characterize Poisson integrals on the Heisenberg motion groups. We also give a generalization of the Segal-Bargmann transform on Heisenberg motion groups.

3.1. Gutzmer's formula on compact, connected Lie groups.

Let \underline{k} and \underline{g} be the Lie algebras of a compact, connected Lie group K and its complexification G . Then we can write $\underline{g} = \underline{k} + \underline{p}$ where $\underline{p} = i\underline{k}$ and any element $g \in G$ can be written in the form $g = k \exp iH$ for some $k \in K$, $H \in \underline{k}$. If \underline{h} is a maximal, abelian subalgebra of \underline{k} and $\underline{a} = i\underline{h}$ then every element of \underline{p} is conjugate under K to an element of \underline{a} . Thus each $g \in G$ can be written (non-uniquely) in the form $g = k_1 \exp(iH)k_2$ for $k_1, k_2 \in K$ and $H \in \underline{h}$. If $k_1 \exp(iH_1)k'_1 = k_2 \exp(iH_2)k'_2$, then there exists $w \in W$, the Weyl group with respect to \underline{h} , such that $H_1 = w \cdot H_2$ where \cdot denotes the action of the Weyl group on \underline{h} . Since K is compact, there exists an $\text{Ad}-K$ -invariant inner product on \underline{k} , and hence on \underline{g} . Let $|\cdot|$ denote the norm with respect to the said inner product. Then we have the following Gutzmer's formula by Lassalle (see [8]).

Theorem 3.1. *Let f be a holomorphic function in $K \exp(i\Omega_r)K \subseteq G$ where $\Omega_r = \{H \in \underline{k} : |H| < r\}$. Then we have*

$$\int_K \int_K |f(k_1 \exp iHk_2)|^2 dk_1 dk_2 = \sum_{\pi \in \hat{K}} \|\hat{f}(\pi)\|_{HS}^2 \chi_\pi(\exp 2iH)$$

where $H \in \Omega_r$ and $\hat{f}(\pi)$ is the operator-valued Fourier transform of f at π defined by $\hat{f}(\pi) = \int_K f(k) \pi(k^{-1}) dk$.

For the proof of above, see [8].

3.2. The Hermite and special Hermite functions.

Here we collect relevant information about Hermite and special Hermite functions. We closely follow the notation used in [15] and we refer the reader to the same for more details.

For every $\lambda \neq 0$, the Schrödinger representation π_λ of the Heisenberg group \mathbb{H}^n on $L^2(\mathbb{R}^n)$ is defined by

$$\pi_\lambda(x, u, t)f(\xi) = e^{i\lambda t} e^{i\lambda(x \cdot \xi + \frac{1}{2}x \cdot u)} f(\xi + u)$$

where $f \in L^2(\mathbb{R}^n)$. A celebrated theorem of Stone-von Neumann says that up to unitary equivalence these are all the irreducible unitary representations of \mathbb{H}^n that are nontrivial at the center.

Theorem 3.2. *The representations π_λ , $\lambda \neq 0$ are unitary and irreducible. If ρ is an irreducible unitary representation of \mathbb{H}^n on a Hilbert space \mathcal{H} such that $\rho(0, 0, t) = e^{i\lambda t}I$ for some $\lambda \neq 0$, then ρ is unitarily equivalent to π_λ .*

Note that $\pi_\lambda(x, u, t) = e^{i\lambda t}\pi_\lambda(x, u, 0)$. We shall write $\pi_\lambda(x, u, 0)$ as $\pi_\lambda(x, u)$. Let ϕ_α , $\alpha \in \mathbb{N}^n$ be the Hermite functions on \mathbb{R}^n normalized so that their L^2 norms are one. The family $\{\phi_\alpha, \alpha \in \mathbb{N}^n\}$ is an orthonormal basis of $L^2(\mathbb{R}^n)$. For $\lambda \neq 0$, we define the scaled Hermite functions

$$\phi_\alpha^\lambda(x) = |\lambda|^{\frac{n}{4}} \phi_\alpha(|\lambda|^{\frac{1}{2}}x).$$

We also consider

$$\phi_{\alpha\beta}^\lambda(x, u) = (2\pi)^{-\frac{n}{2}} |\lambda|^{\frac{n}{2}} \langle \pi_\lambda(x, u, 0) \phi_\alpha^\lambda, \phi_\beta^\lambda \rangle, \quad \text{for } \alpha, \beta \in \mathbb{N}^n$$

which are essentially the matrix coefficients of π_λ at $(x, u, 0) \in \mathbb{H}^n$. They are the so called special Hermite functions and $\{\phi_{\alpha\beta}^\lambda : \alpha, \beta \in \mathbb{N}^n\}$ is a complete orthonormal system in $L^2(\mathbb{R}^{2n})$.

Note that $\phi_\alpha^\lambda(\xi) = H_\alpha^\lambda(\xi) e^{-\frac{\lambda}{2}|\xi|^2}$ where H_α^λ is a polynomial on \mathbb{R}^n . For $z \in \mathbb{C}^n$, we define $\phi_\alpha^\lambda(z)$ to be $H_\alpha^\lambda(z) e^{-\frac{\lambda}{2}z^2}$ where $z^2 = z \cdot z$. Then for $z, w \in \mathbb{C}^n$ we can define

$$\pi_\lambda(z, w, t) \phi_\alpha^\lambda(\xi) = e^{i\lambda t} e^{i\lambda(z \cdot \xi + \frac{1}{2}z \cdot w)} \phi_\alpha^\lambda(\xi + w).$$

Hence we have

$$\phi_{\alpha\beta}^\lambda(z, w) = (2\pi)^{-\frac{n}{2}} |\lambda|^{\frac{n}{2}} \langle \pi_\lambda(z, w) \phi_\alpha^\lambda, \phi_\beta^\lambda \rangle$$

for $z, w \in \mathbb{C}^n$. An easy calculation shows that

$$(3.1) \quad \langle \pi_\lambda(z, w) \phi_\alpha^\lambda, \phi_\beta^\lambda \rangle = \langle \phi_\alpha^\lambda, \pi_\lambda(-\bar{z}, -\bar{w}) \phi_\beta^\lambda \rangle.$$

Notice that both $\phi_\alpha^\lambda(z)$, $\phi_{\alpha\beta}^\lambda(z, w)$ are holomorphic on \mathbb{C}^n and \mathbb{C}^{2n} respectively.

3.3. Gelfand pairs and the Heisenberg group.

For each $k \in K \subseteq U(n)$, $(x, u, t) \rightarrow (k \cdot (x, u), t)$ is an automorphism of \mathbb{H}^n , because $U(n)$ preserves the symplectic form $x \cdot u - y \cdot v$. If ρ is a representation of \mathbb{H}^n , then using this automorphism we can define another representation ρ_k by $\rho_k(x, u, t) = \rho(k \cdot (x, u), t)$ which coincides with ρ at the center. If we take ρ to be the Schrödinger representation π_λ , then by Stone-von Neumann theorem $(\pi_\lambda)_k$ is unitarily equivalent to π_λ and we have the unitary intertwining operator μ_λ such that

$$(3.2) \quad \pi_\lambda(k \cdot (x, u), t) = \mu_\lambda(k) \pi_\lambda(x, u, t) \mu_\lambda(k)^*.$$

The operator valued function μ_λ can be chosen so that it becomes a unitary representation of K on $L^2(\mathbb{R}^n)$ and is called the metaplectic representation. For each $m > 0$, let \mathcal{P}_m be the linear span of $\{\phi_\alpha : |\alpha| = m\}$. Then each such \mathcal{P}_m is invariant under the action of $\mu_\lambda(k)$ for every $k \in K \subseteq U(n)$. When $K = U(n)$, $\mu_\lambda|_{\mathcal{P}_m}$ is irreducible. When K is a proper compact subgroup of $U(n)$, \mathcal{P}_m need not be irreducible under the action of μ_λ . So it further decomposes into irreducible subspaces. It is known that (K, \mathbb{H}^n) is a Gelfand pair if and only if this action of K on $L^2(\mathbb{R}^n)$ decomposes into irreducible components of multiplicity one (see [1]).

Let L_m^{n-1} be the Laguerre polynomials of type $(n-1)$ and define Laguerre functions by

$$\varphi_m^\lambda(x, u) = L_m^{n-1} \left(\frac{|\lambda|}{2} (|x|^2 + |u|^2) \right) e^{-\frac{|\lambda|}{4} (|x|^2 + |u|^2)}.$$

Then it is known that

$$\varphi_m^\lambda(x, u) = \sum_{|\alpha|=m} \phi_{\alpha\alpha}^\lambda(x, u)$$

and $e_m^\lambda(x, u, t) = \frac{1}{\dim \mathcal{P}_m} e^{i\lambda t} \varphi_m^\lambda(x, u)$ is a $U(n)$ -spherical function. Let $\mathcal{P}_m = \bigoplus_{a=1}^{A_m} \mathcal{P}_{ma}$ be the decomposition of \mathcal{P}_m into K -irreducible subspaces. Then

$$e_{ma}^\lambda(x, u, t) = \frac{1}{\dim \mathcal{P}_{ma}} e^{i\lambda t} \varphi_{ma}^\lambda(x, u) = \frac{1}{\dim \mathcal{P}_{ma}} \sum_{b=1}^{B_a} \langle \pi_\lambda(x, u, t) \phi_{ma}^b, \phi_{ma}^b \rangle$$

is a K -spherical function where $\{\phi_{ma}^b : b = 1 \cdots B_a\}$ is an orthonormal basis for \mathcal{P}_{ma} such that $\{\phi_{ma}^b : b = 1 \cdots B_a, a = 1 \cdots A_m\}$ is an orthonormal basis for \mathcal{P}_m . For more on Gelfand pairs and spherical functions on \mathbb{H}^n see [1].

3.4. Gutzmer's formula for K -special Hermite functions.

Let us write $\{\phi_{ma}^b : b = 1 \cdots B_a, a = 1 \cdots A_m\}$ as $\{\psi_\alpha : \alpha \in \mathbb{N}^n\}$ such that for each m , $\{\phi_{ma}^b : b = 1 \cdots B_a, a = 1 \cdots A_m\}$ are the ones which occur as ψ_α for $|\alpha| = m$. For $\lambda \neq 0$, we define

$$\psi_\alpha^\lambda(x) = |\lambda|^{\frac{n}{4}} \psi_\alpha(|\lambda|^{\frac{1}{2}} x).$$

Consider

$$\psi_{\alpha\beta}^\lambda(x, u) = (2\pi)^{-\frac{n}{2}} |\lambda|^{\frac{n}{2}} \langle \pi_\lambda(x, u) \psi_\alpha^\lambda, \psi_\beta^\lambda \rangle$$

and we call them K -special Hermite functions. It is easy to see that $\{\psi_{\alpha\beta}^\lambda : \alpha, \beta \in \mathbb{N}^n\}$ is a complete orthonormal system in $L^2(\mathbb{R}^{2n})$. Since each ψ_α is a finite linear combination of ϕ_α 's, both ψ_α^λ and $\psi_{\alpha\beta}^\lambda$ extend as holomorphic functions to \mathbb{C}^n and \mathbb{C}^{2n} respectively for each $\alpha, \beta \in \mathbb{N}^n$. We also note that the action of $K \subseteq U(n)$ on \mathbb{R}^{2n} naturally extends to an action of G on \mathbb{C}^{2n} .

Theorem 3.3. *For a function $F \in L^2(\mathbb{R}^{2n})$ having a holomorphic extension to \mathbb{C}^{2n} , we have*

$$\begin{aligned} & \int_K \int_{\mathbb{R}^{2n}} |F(k \cdot (x + iy, u + iv))|^2 e^{\lambda(u \cdot y - v \cdot x)} dx du dk \\ &= \sum_{m=0}^{\infty} \sum_{a=1}^{A_m} (\dim \mathcal{P}_{ma})^{-1} \varphi_{ma}^\lambda(2iy, 2iv) \|F *_\lambda \varphi_{ma}^\lambda\|^2, \end{aligned}$$

whenever either of them is finite.

Proof. First we want to prove that $\psi_{\alpha\beta}^\lambda$'s are orthogonal under the inner product

$$\langle F, G \rangle = \int_K \int_{\mathbb{R}^{2n}} F(k \cdot (x + iy, u + iv)) \overline{G(k \cdot (x + iy, u + iv))} e^{\lambda(u \cdot y - v \cdot x)} dx du dk$$

for $F, G \in L^2(\mathbb{R}^n)$ which have a holomorphic extension to \mathbb{C}^{2n} . So, we consider

$$\begin{aligned} & \int_K \int_{\mathbb{R}^{2n}} \psi_{\alpha\beta}^\lambda(k \cdot (x + iy, u + iv)) \overline{\psi_{\mu\nu}^\lambda(k \cdot (x + iy, u + iv))} e^{\lambda(u \cdot y - v \cdot x)} dx du dk \\ &= (2\pi)^{-n} |\lambda|^n \int_K \int_{\mathbb{R}^{2n}} \langle \pi_\lambda(k \cdot (x, u)) \psi_\alpha^\lambda, \pi_\lambda(k \cdot (iy, iv)) \psi_\beta^\lambda \rangle \overline{\langle \pi_\lambda(k \cdot (x, u)) \psi_\mu^\lambda, \pi_\lambda(k \cdot (iy, iv)) \psi_\nu^\lambda \rangle} dx du dk \end{aligned}$$

by (3.1). Expanding $\pi_\lambda(k \cdot (x, u)) \psi_\alpha^\lambda$ in terms of ψ_ρ^λ , $\pi_\lambda(k \cdot (x, u)) \psi_\mu^\lambda$ in terms of ψ_σ^λ and using the self-adjointness of $\pi_\lambda(k \cdot (iy, iv))$ the above equals

$$\begin{aligned} & \sum_{\rho, \sigma \in \mathbb{N}^n} \int_K \langle \pi_\lambda(k \cdot (iy, iv)) \psi_\rho^\lambda, \psi_\beta^\lambda \rangle \overline{\langle \pi_\lambda(k \cdot (iy, iv)) \psi_\sigma^\lambda, \psi_\nu^\lambda \rangle} \left(\int_{\mathbb{R}^{2n}} \psi_{\alpha\rho}^\lambda(k \cdot (x, u)) \overline{\psi_{\mu\sigma}^\lambda(k \cdot (x, u))} dx du \right) dk \\ &= \delta_{\alpha, \mu} \int_K \langle \pi_\lambda(k \cdot (2iy, 2iv)) \psi_\nu^\lambda, \psi_\beta^\lambda \rangle dk, \end{aligned}$$

δ being the Kronecker delta. Then using (3.2), expanding $\mu_\lambda(k^{-1}) \psi_\nu^\lambda$ in terms of ψ_γ^λ , for $\gamma \in \mathcal{P}_{ma}$, $\mu_\lambda(k^{-1}) \psi_\beta^\lambda$ in terms of ψ_δ^λ for $\delta \in \mathcal{P}_{lb}$ and using Schur's orthogonality relations we get that the above equals

$$\begin{aligned} & \delta_{\alpha, \mu} \int_K \langle \pi_\lambda(2iy, 2iv) \mu_\lambda(k^{-1}) \psi_\nu^\lambda, \mu_\lambda(k^{-1}) \psi_\beta^\lambda \rangle dk \\ &= \delta_{\alpha, \mu} \sum_{\gamma \in \mathcal{P}_{ma}} \sum_{\delta \in \mathcal{P}_{lb}} \left(\int_K \eta_{\gamma\nu}(k^{-1}) \overline{\eta_{\delta\beta}(k^{-1})} dk \right) \langle \pi_\lambda(2iy, 2iv) \psi_\gamma^\lambda, \psi_\delta^\lambda \rangle \\ &= \delta_{\alpha, \mu} \delta_{\beta, \nu} \dim \mathcal{P}_{ma}^{-1} \varphi_{ma}^\lambda(2iy, 2iv) \end{aligned}$$

where \mathcal{P}_{ma} and \mathcal{P}_{lb} are the ones which contain ψ_ν^λ and ψ_β^λ respectively and $\eta_{\gamma\nu}$'s are the matrix coefficients of μ_λ .

Then for a function F as in the statement of the theorem we have

$$\begin{aligned} & \int_K \int_{\mathbb{R}^{2n}} |F(k \cdot (x + iy, u + iv))|^2 e^{\lambda(u \cdot y - v \cdot x)} dx du dk \\ &= \sum_{m=0}^{\infty} \sum_{a=1}^{A_m} (\dim \mathcal{P}_{ma})^{-1} \varphi_{ma}^\lambda(2iy, 2iv) \left(\sum_{\alpha \in \mathbb{N}^n} \sum_{\beta \in \mathcal{P}_{ma}} |\langle F, \phi_{\alpha\beta}^\lambda \rangle|^2 \right). \end{aligned}$$

Now, it is easy to see from standard arguments (see [15] for details) that $\|F *_{\lambda} \varphi_{ma}^{\lambda}\|^2 = \sum_{\alpha \in \mathbb{N}^n} \sum_{\beta \in \mathcal{P}_{ma}} |\langle F, \psi_{\alpha\beta}^{\lambda} \rangle|^2$ and hence the theorem follows. \square

3.5. Poisson integrals for the Laplacian on Heisenberg motion groups.

Proposition 4.1 of [10] gives that for $f \in L^2(\mathbb{H}^n)$,

$$f(x, u, t) = (2\pi)^{-n} \sum_{m=0}^{\infty} \sum_{a=1}^{A_m} \int_{\mathbb{R}} f * e_{ma}^{\lambda}(x, u, t) |\lambda|^n d\lambda.$$

Recall that the Laplacian Δ on \mathbb{HM} is given by $\Delta = -\Delta_{\mathbb{H}^n} - \Delta_K$ and for suitable functions f on \mathbb{HM} , a function f^{λ} is defined on $\mathbb{R}^{2n} \times K$ by

$$f^{\lambda}(x, u, k) = \int_{\mathbb{R}} f(x, u, t, k) e^{i\lambda t} dt.$$

For $f \in L^2(\mathbb{HM})$, we have the expansion

$$f(x, u, t, k) = \sum_{\pi \in \hat{K}} d_{\pi} \sum_{i,j=1}^{d_{\pi}} f_{ij}^{\pi}(x, u, t) \phi_{ij}^{\pi}(k)$$

where $f_{ij}^{\pi}(x, u, t) = \int_K f(x, u, t, k) \overline{\phi_{ij}^{\pi}(k)} dk$. Then it is easy to see that

$$\begin{aligned} & e^{-q\Delta^{\frac{1}{2}}} f(x, u, t, k) \\ &= (2\pi)^{-n} \sum_{\pi \in \hat{K}} d_{\pi} \sum_{i,j=1}^{d_{\pi}} \left(\int_{\mathbb{R}} \sum_{m=0}^{\infty} \sum_{a=1}^{A_m} e^{-q((2m+n)|\lambda| + \lambda^2 + \lambda_{\pi})^{\frac{1}{2}}} (f_{ij}^{\pi})^{\lambda} *_{\lambda} \varphi_{ma}^{\lambda}(x, u) e^{i\lambda t} |\lambda|^n d\lambda \right) \\ & \quad \phi_{ij}^{\pi}(k). \end{aligned}$$

We have the following (almost) characterization of the Poisson integrals. Let $\Omega_{p,p'}$ be the domain in $\mathbb{C}^{2n+1} \times G$ defined by $\{(z, w, \tau, g) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \times G : |Im(z, w)| < p, |H| < p' \text{ where } g = k_1(\exp iH)k_2, k_1, k_2 \in K, H \in \underline{h}\}$. Notice that the domain $\Omega_{p,p'}$ is well defined since $|\cdot|$ is invariant under the Weyl group action.

Theorem 3.4. *Let $f \in L^1 \cap L^2(\mathbb{HM})$ be such that f^{λ} is compactly supported as a function of λ . Then there exists a constant N such that for each $0 < p < q$,*

$h = e^{-q\Delta^{\frac{1}{2}}} f$ extends to a holomorphic function on the domain $\Omega_{\frac{p}{2}, \frac{p}{N\sqrt{2}}}$ and

$$\begin{aligned} & \int_K \int_K \int_{|Im(z,w)|=r} \int_{\mathbb{H}^n} |h(X \cdot (z, w, \tau, k_1 \exp(iH)k_2))|^2 dX d\mu_r dk_1 dk_2 \\ &= \sum_{\pi \in \widehat{K}} d_\pi \chi_\pi(\exp 2iH) \sum_{m=0}^{\infty} \sum_{a=1}^{A_m} (\dim \mathcal{P}_m)^{-1} \int_{\mathbb{R}} L_m^{n-1}(-2\lambda r^2) e^{\lambda r^2} e^{2\lambda Im\tau} \\ & \quad e^{-2q((2m+n)|\lambda|+\lambda^2+\lambda_\pi)^{\frac{1}{2}}} \sum_{i,j=1}^{d_\pi} \|(f_{ij}^\pi)^\lambda *_{\lambda} \varphi_{ma}^\lambda\|^2 d\lambda \end{aligned}$$

where μ_r is the normalized surface area measure on the sphere $\{|Im(z, w)| = r\} \subseteq \mathbb{R}^{2n}$ for $r < \frac{p}{2}$ and L_m^{n-1} are the Laguerre polynomials of type $(n-1)$.

Conversely, there exists a fixed constant V such that if h is a holomorphic function on the domain $\Omega_{q, \frac{2q}{V}}$, h^λ is compactly supported as a function of λ and for each $r < q$

$$\int_K \int_K \int_{|Im(z,w)|=r} \int_{\mathbb{H}^n} |h(X \cdot (z, w, \tau, k_1 \exp(iH)k_2))|^2 dX d\mu_r dk_1 dk_2 < \infty,$$

then for every $p < q$, there exists $f \in L^2(\mathbb{H}M)$ such that $h = e^{-p\Delta^{\frac{1}{2}}} f$.

Proof. First, we prove the holomorphicity of $e^{-q\Delta^{\frac{1}{2}}} f$ on $\Omega_{\frac{p}{2}, \frac{p}{N\sqrt{2}}}$ for $0 < p < q$ by proving uniform convergence of the same on compact subsets. So, we consider a compact subset $M \subseteq \Omega_{\frac{p}{2}, \frac{p}{N\sqrt{2}}}$.

Since

$$\begin{aligned} e^{-q((2m+n)|\lambda|+\lambda^2+\lambda_\pi)^{\frac{1}{2}}} &\leq e^{-\frac{q\sqrt{\lambda_\pi}}{\sqrt{2}}} e^{-\frac{q}{\sqrt{2}}((2m+n)|\lambda|+\lambda^2)^{\frac{1}{2}}} \\ &\leq e^{-\frac{q\sqrt{\lambda_\pi}}{\sqrt{2}}} e^{-\frac{q}{2}((2m+n)|\lambda|)^{\frac{1}{2}}} e^{-\frac{q|\lambda|}{2}}, \end{aligned}$$

for $(z, w, \tau, g) = (x + iy, u + iv, t + is, ke^{iH}) \in M \subset \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \times G$, we have

$$\begin{aligned} & \left| e^{-q\Delta^{\frac{1}{2}}} f(z, w, \tau, g) \right| \\ &\leq C \sum_{\pi \in \widehat{K}} d_\pi \sum_{i,j=1}^{d_\pi} |\phi_{ij}^\pi(g)| e^{-\frac{q\sqrt{\lambda_\pi}}{\sqrt{2}}} \int_{\mathbb{R}} \left(\sum_{m=0}^{\infty} \sum_{a=1}^{A_m} e^{-\frac{q}{2}((2m+n)|\lambda|)^{\frac{1}{2}}} |(f_{ij}^\pi)^\lambda *_{\lambda} \varphi_{ma}^\lambda(z, w)| \right) \\ & \quad e^{|\lambda|(|s|-\frac{q}{2})} |\lambda|^n d\lambda. \end{aligned}$$

Now, it can be seen that for a fixed λ ,

$$|\varphi_{ma}^\lambda(z, w)|^2 \leq C \dim \mathcal{P}_{ma} e^{\lambda(u \cdot y - v \cdot x)} \varphi_{ma}^\lambda(2iy, 2iv).$$

It follows that

$$|(f_{ij}^\pi)^\lambda *_\lambda \varphi_{ma}^\lambda(z, w)| \leq e^{\frac{\lambda}{2}(u \cdot y - v \cdot x)} \|(f_{ij}^\pi)^\lambda\|_1 (\dim \mathcal{P}_{ma})^{\frac{1}{2}} (\varphi_{ma}^\lambda(2iy, 2iv))^{\frac{1}{2}}.$$

So, we get that

$$\begin{aligned} & \sum_{m=0}^{\infty} \sum_{a=1}^{A_m} e^{-\frac{q}{2}((2m+n)|\lambda|)^{\frac{1}{2}}} |(f_{ij}^\pi)^\lambda *_\lambda \varphi_{ma}^\lambda(z, w)| \\ & \leq e^{\frac{\lambda}{2}(u \cdot y - v \cdot x)} \|(f_{ij}^\pi)^\lambda\|_1 \sum_{m=0}^{\infty} \sum_{a=1}^{A_m} e^{-\frac{q}{2}((2m+n)|\lambda|)^{\frac{1}{2}}} (\dim \mathcal{P}_{ma})^{\frac{1}{2}} (\varphi_{ma}^\lambda(2iy, 2iv))^{\frac{1}{2}}. \end{aligned}$$

Applying Cauchy-Schwarz inequality to the above and noting that $\varphi_m^\lambda(2iy, 2iv) = \sum_{a=1}^{A_m} \varphi_{ma}^\lambda(2iy, 2iv)$ (see [10] for details) and $\dim \mathcal{P}_m = \sum_{a=1}^{A_m} \dim \mathcal{P}_{ma} = \frac{(m+n-1)!}{m!(n-1)!}$ we get that

$$\begin{aligned} & \sum_{m=0}^{\infty} \sum_{a=1}^{A_m} e^{-\frac{q}{2}((2m+n)|\lambda|)^{\frac{1}{2}}} |(f_{ij}^\pi)^\lambda *_\lambda \varphi_{ma}^\lambda(z, w)| \\ & \leq e^{\frac{\lambda}{2}(u \cdot y - v \cdot x)} \|(f_{ij}^\pi)^\lambda\|_1 \sum_{m=0}^{\infty} e^{-\frac{q}{2}((2m+n)|\lambda|)^{\frac{1}{2}}} \left(\frac{(m+n-1)!}{m!(n-1)!} \right)^{\frac{1}{2}} (\varphi_m^\lambda(2iy, 2iv))^{\frac{1}{2}}. \end{aligned}$$

As in the proof of Theorem 5.1 of [14], for any fixed (y, v) with $|y|^2 + |v|^2 \leq r^2 < \frac{p^2}{4} < \frac{q^2}{4}$, the above series is bounded by a constant times

$$\sum_{m=0}^{\infty} m^{\frac{n-1}{2}} m^{\frac{n-1}{4} - \frac{1}{8}} e^{-((2m+n)|\lambda|)^{\frac{1}{2}} (\frac{q}{2} - r)}$$

which certainly converges if $r < \frac{p}{2} < \frac{q}{2}$. Moreover, using the fact that $\|(f_{ij}^\pi)^\lambda\|_1 \leq \|f\|_1$ and f^λ is compactly supported as a function of λ , we can conclude that

$$\begin{aligned} & \left| e^{-q\Delta^{\frac{1}{2}}} f(z, w, \tau, g) \right| \\ & \leq C \sum_{\pi \in \hat{K}} d_\pi \sum_{i,j=1}^{d_\pi} |\phi_{ij}^\pi(g)| e^{-\frac{q\sqrt{\lambda_\pi}}{\sqrt{2}}}. \end{aligned}$$

For $g = ke^{iH}$, we have $\phi_{ij}^\pi(ke^{iH}) = \sum_{l=1}^{d_\pi} \phi_{il}^\pi(k) \phi_{lj}^\pi(e^{iH})$. Since $\pi(k)$ is unitary for $k \in K$

and $\pi(e^{iH})$ is self-adjoint for $H \in \underline{h}$, it follows that $\sum_{l=1}^{d_\pi} |\phi_{il}^\pi(k)|^2 = 1$ and

$$\begin{aligned} \sum_{l,j=1}^{d_\pi} |\phi_{lj}^\pi(e^{iH})|^2 &= \sum_{l,j=1}^{d_\pi} \langle \pi(e^{iH})e_l, e_j \rangle \overline{\langle \pi(e^{iH})e_l, e_j \rangle} \\ &= \sum_{l,j=1}^{d_\pi} \langle \pi(e^{iH})e_l, e_j \rangle \langle e_j, \pi(e^{iH})e_l \rangle \\ &= \sum_{l=1}^{d_\pi} \langle \pi(e^{iH})e_l, \pi(e^{iH})e_l \rangle \\ &= \chi_\pi(\exp 2iH) \end{aligned}$$

where $e_1, e_2, \dots, e_{d_\pi}$ is an orthonormal basis of the Hilbert space \mathcal{H}_π on which π acts. Now, using Cauchy-Schwarz inequality we get that

$$\begin{aligned} &\left| e^{-q\Delta^{\frac{1}{2}}} f(z, w, \tau, g) \right| \\ &\leq C \sum_{\pi \in \hat{K}} d_\pi^{\frac{5}{2}} (\chi_\pi(\exp 2iH))^{\frac{1}{2}} e^{-\frac{q\sqrt{\lambda_\pi}}{\sqrt{2}}}. \end{aligned}$$

From Lemma 6 and 7 of [4] we know that there exist constants A, B, C and M such that $\lambda_\pi \geq A|\mu|^2$, $d_\pi \leq B(1 + |\mu|^C)$ and $|\chi_\pi(\exp iY)| \leq d_\pi e^{M|Y||\mu|}$ where μ is the highest weight of π . Hence we have

$$|\chi_\pi(\exp 2iH)| \leq d_\pi e^{2M|H||\mu|} \leq d_\pi e^{2N|H|\sqrt{\lambda_\pi}}$$

where $N = \frac{M}{\sqrt{A}}$. It follows that

$$\begin{aligned} &\left| e^{-q\Delta^{\frac{1}{2}}} f(z, w, \tau, g) \right| \\ &\leq C \sum_{\pi \in \hat{K}} B^3 \left(1 + \left(\frac{\lambda_\pi}{A} \right)^{\frac{C}{2}} \right)^3 e^{\sqrt{\lambda_\pi} (N|H| - \frac{q}{\sqrt{2}})} \end{aligned}$$

which is finite as long as $|H| < \frac{q}{N\sqrt{2}}$. Hence we have proved that $e^{-q\Delta^{\frac{1}{2}}} f$ extends to a holomorphic function on the domain $\Omega_{\frac{p}{2}, \frac{p}{N\sqrt{2}}}$ for $p < q$.

Now, we prove the equality in Theorem 3.4. It should be noted that the domain $\Omega_{\frac{p}{2}, \frac{p}{N\sqrt{2}}}$ is invariant under left translation by the Heisenberg motion group \mathbb{HM} . For $X = (x', u', t', k') \in \mathbb{HM}$, $(z, w, \tau, g) = (x + iy, u + iv, t + is, k_1 \exp(iH)k_2) \in \Omega_{\frac{p}{2}, \frac{p}{N\sqrt{2}}} \subseteq \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \times G$ and a function F holomorphic on $\Omega_{\frac{p}{2}, \frac{p}{N\sqrt{2}}}$, by Gutzmer's formula on K we have

$$\begin{aligned} & \int_K \int_K \int_{|Im(z, w)|=r} \int_{\mathbb{H}^n} |F(X \cdot (z, w, \tau, k_1 \exp(iH)k_2))|^2 dX d\mu_r dk_1 dk_2 \\ &= \sum_{\pi \in \widehat{K}} d_\pi \sum_{i, j=1}^{d_\pi} \int_{|Im(z, w)|=r} \int_{\mathbb{R}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |F_{ij}^\pi(x' + iy, u' + iv, t' + i(s + \frac{1}{2}(u' \cdot y - v \cdot x')))|^2 \\ & \quad \chi_\pi(\exp 2iH) dx' du' dt' d\mu_r. \end{aligned}$$

It is easy to see that $\frac{1}{\dim \mathcal{P}_{ma}} \int_{U(n)} \varphi_{ma}^\lambda(k \cdot (x, u)) dk$ is a $U(n)$ -spherical function. So it is obvious that $\frac{1}{\dim \mathcal{P}_{ma}} \int_{U(n)} \varphi_{ma}^\lambda(k \cdot (x, u)) dk = \frac{1}{\dim \mathcal{P}_m} \varphi_m^\lambda(x, u)$. By analytic continuation on both sides we get

$$\frac{1}{\dim \mathcal{P}_{ma}} \int_{U(n)} \varphi_{ma}^\lambda(k \cdot (2iy, 2iv)) dk = \frac{1}{\dim \mathcal{P}_m} \varphi_m^\lambda(2iy, 2iv).$$

Hence it follows that the integral over $U(n)$ can be seen as an integral over the sphere $|y|^2 + |v|^2 = r^2$ such that

$$\frac{1}{\dim \mathcal{P}_{ma}} \int_{|y|^2 + |v|^2 = r^2} \varphi_{ma}^\lambda(2iy, 2iv) d\mu_r = \frac{1}{\dim \mathcal{P}_m} L_m^{n-1}(-2\lambda r^2) e^{\lambda r^2}.$$

So, from Theorem 3.3 we have

$$\begin{aligned} & \int_{|y|^2 + |v|^2 = r^2} \int_{\mathbb{R}^{2n}} |(F_{ij}^\pi)^\lambda(x + iy, u + iv)|^2 e^{\lambda(u \cdot y - v \cdot x)} dx du d\mu_r \\ &= \sum_{m=0}^{\infty} \sum_{a=1}^{A_m} (\dim \mathcal{P}_m)^{-1} L_m^{n-1}(-2\lambda r^2) e^{\lambda r^2} \|(F_{ij}^\pi)^\lambda *_{\lambda} \varphi_{ma}^\lambda\|^2 \end{aligned}$$

It follows that

$$\begin{aligned}
& \int_K \int_K \int_{|Im(z,w)|=r} \int_{\mathbb{H}^n} |F(X \cdot (z, w, \tau, k_1 \exp(iH)k_2))|^2 dX d\mu_r dk_1 dk_2 \\
&= \sum_{\pi \in \widehat{K}} d_\pi \chi_\pi(\exp 2iH) \sum_{m=0}^{\infty} \sum_{a=1}^{A_m} (\dim \mathcal{P}_m)^{-1} \int_{\mathbb{R}} L_m^{n-1}(-2\lambda r^2) e^{\lambda r^2} e^{2\lambda s} \\
& \quad \left(\sum_{i,j=1}^{d_\pi} \|(F_{ij}^\pi)^\lambda *_{\lambda} \varphi_{ma}^\lambda\|^2 \right) d\lambda.
\end{aligned}$$

Hence for $h = e^{-q\Delta^{\frac{1}{2}}} f$ we get the first part of Theorem 3.4.

To prove the converse, we first show that for any $0 < \vartheta < \infty$, there exist constants U, V such that

$$(3.3) \quad \int_{|H|=\vartheta} \chi_\pi(\exp 2iH) d\sigma_\vartheta(H) \geq d_\pi U e^{V\vartheta\sqrt{\lambda_\pi}}$$

where $d\sigma_\vartheta(H)$ is the normalized surface measure on the sphere $\{H \in \underline{h} : |H| = \vartheta\} \subseteq \mathbb{R}^m$ and $m = \dim \underline{h}$. If $H \in \underline{a}$, then there exists a non-singular matrix Q and pure-imaginary valued linear forms $\nu_1, \nu_2, \dots, \nu_{d_\pi}$ on \underline{a} such that

$$Q\pi(H)Q^{-1} = \text{diag}(\nu_1(H), \nu_2(H), \dots, \nu_{d_\pi}(H))$$

where $\text{diag}(a_1, a_2, \dots, a_k)$ denotes $k \times k$ diagonal matrix with diagonal entries a_1, a_2, \dots, a_k .

Now, $\nu(H) = i\langle \nu, H \rangle$ where ν is a weight of π . Then

$$\exp(2iQ\pi(H)Q^{-1}) = Q \exp(2i\pi(H))Q^{-1} = \text{diag}(e^{2i\nu_1(H)}, e^{2i\nu_2(H)}, \dots, e^{2i\nu_{d_\pi}(H)}).$$

Hence

$$\begin{aligned}
\chi_\pi(\exp 2iH) &= \text{Tr}(Q \exp(2i\pi(H))Q^{-1}) \\
&= e^{-2\langle \nu_1, H \rangle} + e^{-2\langle \nu_2, H \rangle} + \dots + e^{-2\langle \nu_{d_\pi}, H \rangle} \\
&\geq e^{-2\langle \mu, H \rangle}
\end{aligned}$$

where μ is the highest weight corresponding to π . Integrating the above over $|H| = \vartheta$ we get

$$\begin{aligned} \int_{|H|=\vartheta} \chi_\pi(\exp 2iH) d\sigma_\vartheta(H) &\geq \int_{|H|=\vartheta} e^{-2\langle \mu, H \rangle} d\sigma_\vartheta(H) \\ &= \frac{J_{\frac{m}{2}-1}(2i\vartheta|\mu|)}{(2i\vartheta|\mu|)^{\frac{m}{2}-1}} \\ &\geq U e^{\vartheta|\mu|} \end{aligned}$$

where $J_{\frac{m}{2}-1}$ is the Bessel function of order $(\frac{m}{2} - 1)$. It is known that $\lambda_\pi \approx |\mu|^2$, hence we have

$$(3.4) \quad \int_{|H|=\vartheta} \chi_\pi(\exp 2iH) d\sigma_\vartheta(H) \geq U e^{V\vartheta\sqrt{\lambda_\pi}}$$

for some V . Consider the domain $\Omega_{q, \frac{2q}{V}}$ for this V . Let h be a holomorphic function on the domain $\Omega_{q, \frac{2q}{V}}$ such that h^λ is compactly supported as a function of λ and for $r < q$,

$$\int_K \int_K \int_{|Im(z,w)|=r} \int_{\mathbb{H}^n} |h(X \cdot (z, w, \tau, k_1 \exp(iH)k_2))|^2 dX d\mu_r dk_1 dk_2 < \infty.$$

So, as before it follows that

$$\begin{aligned} \sum_{\pi \in \hat{K}} d_\pi \chi_\pi(\exp 2iH) \sum_{m=0}^{\infty} \sum_{a=1}^{A_m} (\dim \mathcal{P}_m)^{-1} \int_{\mathbb{R}} L_m^{n-1}(-2\lambda r^2) e^{\lambda r^2} e^{2\lambda s} \\ \left(\sum_{i,j=1}^{d_\pi} \|(h_{ij}^\pi)^\lambda *_{\lambda} \varphi_{ma}^\lambda\|^2 \right) d\lambda < \infty \text{ for } r < q. \end{aligned}$$

Integrating over $|H| = \vartheta$ for $\vartheta < \frac{2q}{V}$ and using (3.4), it also follows that

$$\begin{aligned} \sum_{\pi \in \hat{K}} d_\pi e^{V\vartheta\sqrt{\lambda_\pi}} \sum_{m=0}^{\infty} \sum_{a=1}^{A_m} (\dim \mathcal{P}_m)^{-1} \int_{\mathbb{R}} L_m^{n-1}(-2\lambda r^2) e^{\lambda r^2} e^{2\lambda s} \\ \left(\sum_{i,j=1}^{d_\pi} \|(h_{ij}^\pi)^\lambda *_{\lambda} \varphi_{ma}^\lambda\|^2 \right) d\lambda < \infty \text{ for } r < q. \end{aligned}$$

Now, Perron's formula (Theorem 8.22.3 of [13]) gives

$$L_m^\alpha(\zeta) = \frac{1}{2} \pi^{-\frac{1}{2}} e^{\frac{\zeta}{2}} (-\zeta)^{-\frac{\alpha}{2}-\frac{1}{4}} m^{\frac{\alpha}{2}-\frac{1}{4}} e^{2(-m\zeta)^{\frac{1}{2}}} \left(1 + O(m^{-\frac{1}{2}}) \right)$$

valid for ζ in the complex plane cut along the positive real axis. Note that we require the formula when $\zeta < 0$. So, using the fact that $\dim \mathcal{P}_m = \frac{(m+n-1)!}{m!(n-1)!}$ and Perron's formula we get that

$$\sum_{\pi \in \widehat{K}} d_\pi e^{V\vartheta\sqrt{\lambda_\pi}} \sum_{m=0}^{\infty} \int_{\mathbb{R}} |\lambda|^{2n} e^{2\varsigma((2m+n)|\lambda|)^{\frac{1}{2}}} e^{2\lambda s} \sum_{i,j=1}^{d_\pi} \|(h_{ij}^\pi)^\lambda *_{\lambda} \varphi_{ma}^\lambda\|^2 d\lambda < \infty$$

for $\varsigma < r < q$ and $\vartheta < \frac{2q}{V}$. For $p < q$, defining $(f_{ij}^\pi)^\lambda$ by $(f_{ij}^\pi)^\lambda = e^{2p((2m+1)|\lambda|+\lambda^2+\lambda_\pi)^{\frac{1}{2}}}$ $(h_{ij}^\pi)^\lambda$ and using the inequality

$$e^{2p((2m+1)|\lambda|+\lambda^2+\lambda_\pi)^{\frac{1}{2}}} \leq e^{2p((2m+1)|\lambda|)^{\frac{1}{2}}} e^{2p|\lambda|} e^{2p\sqrt{\lambda_\pi}}$$

we obtain

$$f(x, u, t, k) = \sum_{\pi \in \widehat{K}} d_\pi \sum_{i,j=1}^{d_\pi} f_{ij}^\pi(x, u, t) \phi_{ij}^\pi(k) \in L^2(\mathbb{H}\mathbb{M})$$

and $h = e^{-p\Delta^{\frac{1}{2}}} f$. □

3.6. A generalization of the Segal-Bargmann transform.

In [4] Brian C. Hall proved the following generalizations of the Segal-Bargmann transforms for \mathbb{R}^n and compact Lie groups :

Theorem 3.5.

(I) Let μ be any measurable function on \mathbb{R}^n such that

- μ is strictly positive and locally bounded away from zero,
- $\forall x \in \mathbb{R}^n, \sigma(x) = \int_{\mathbb{R}^n} e^{2x \cdot y} \mu(y) dy < \infty$.

Define, for $z \in \mathbb{C}^n$

$$\psi(z) = \int_{\mathbb{R}^n} \frac{e^{ia(y)}}{\sqrt{\sigma(y)}} e^{-iy \cdot z} dy,$$

where a is a real valued measurable function on \mathbb{R}^n . Then the mapping $C_\psi : L^2(\mathbb{R}^n) \rightarrow \mathcal{O}(\mathbb{C}^n)$ defined by

$$C_\psi(z) = \int_{\mathbb{R}^n} f(x) \psi(z - x) dx$$

is an isometric isomorphism of $L^2(\mathbb{R}^n)$ onto $\mathcal{O}(\mathbb{C}^n) \cap L^2(\mathbb{C}^n, dx \mu(y) dy)$.

(II) Let K be a compact Lie group and G be its complexification. Let ν be a measure on G such that

- ν is bi- K -invariant,
- ν is given by a positive density which is locally bounded away from zero,
- For each irreducible representation π of K , analytically continued to G ,

$$\delta(\pi) = \frac{1}{\dim V_\pi} \int_G \|\pi(g^{-1})\|^2 d\nu(g) < \infty.$$

Define $\tau(g) = \sum_{\pi \in \widehat{K}} \frac{d_\pi}{\sqrt{\delta(\pi)}} \text{Tr}(\pi(g^{-1})U_\pi)$ where $g \in G$ and U_π 's are arbitrary unitary matrices. Then the mapping

$$C_\tau f(g) = \int_K f(k) \tau(k^{-1}g) dk$$

is an isometric isomorphism of $L^2(K)$ onto $\mathcal{O}(G) \cap L^2(G, d\nu(w))$.

In a similar fashion, we prove a generalization of Theorem 2.2 for \mathbb{HM} . For each non-zero $\lambda \in \mathbb{R}$, let W_λ be a K -invariant measurable function on \mathbb{R}^{2n} such that it satisfies the following conditions :

- W_λ is strictly positive and locally bounded away from zero uniformly in λ ,
- For each $0 \leq m < \infty$ and $1 \leq a \leq A_m$,

$$\sigma_{m,a,\lambda} = \int_{\mathbb{R}^{2n}} \varphi_{ma}^\lambda(2iy, 2iv) W_\lambda(y, v) dy dv < \infty.$$

For $x, u \in \mathbb{R}^n$, define

$$p^\lambda(x, u) = \left(\frac{2\pi}{|\lambda|} \right)^{-\frac{n}{2}} \sum_{m=0}^{\infty} \sum_{a=1}^{A_m} C_{m,a,\lambda} (\sigma_{m,a,\lambda})^{-\frac{1}{2}} (\dim \mathcal{P}_{ma})^{\frac{1}{2}} \varphi_{ma}^\lambda(x, u),$$

where $|C_{m,a,\lambda}| = 1$. For $(x, u, t) \in \mathbb{H}^n$, consider

$$p(x, u, t) = (4\pi)^{-n} \int_{\mathbb{R}} e^{-i\lambda t} p^\lambda(x, u) e^{-\lambda^2} d\lambda.$$

Next, let ν , δ and τ be as in Theorem 3.5 (II). Also define $\psi(X, k) = p(X)\tau(k)$ for $X \in \mathbb{H}^n$, $k \in K$. Since $\sigma_{m,a,\lambda}$ has exponential growth, it can be proved in a manner similar to Theorem 2.2 and Theorem 3.4 that ψ extends to a holomorphic function

on $\mathbb{H}^n \times G$. Let $\mathcal{A}^\lambda(\mathbb{C}^{2n} \times G)$ be the weighted Bergman space $\mathcal{A}^\lambda(\mathbb{C}^{2n} \times G) = \{F$
entire

$$\text{on } \mathbb{C}^{2n} \times G : \int_G \int_{\mathbb{C}^{2n}} |F(x + iy, u + iv, g)|^2 W_\lambda(y, v) e^{\lambda(u \cdot y - v \cdot x)} dx du dy dv d\nu(g) < \infty\}.$$

Then using Theorem 3.3 it is easy to prove the following analogue of Theorem 3.5 for \mathbb{HM} following the methods in Theorem 2.2.

Theorem 3.6. *The mapping*

$$C_\psi f(Z, g) = \int_{\mathbb{HM}} f(X, k) \phi((X, k)^{-1}(Z, g)) dX dk$$

is an isometric isomorphism of $L^2(\mathbb{HM})$ onto $\int_{\mathbb{R}^}^\oplus \mathcal{A}^\lambda(\mathbb{C}^{2n} \times G) e^{2\lambda^2} d\lambda$.*

4. PLANCHEREL THEOREM AND COMPLEXIFIED REPRESENTATIONS

In this section, we first state and prove the Plancherel theorem for Heisenberg motion groups. Thereby, we also list all the irreducible unitary representations of \mathbb{HM} which occur in the Plancherel theorem. We then use these representations to prove a Paley-Wiener type theorem, which is inspired by Theorem 3.1 of [3].

4.1. Representations of \mathbb{HM} and Plancherel theorem.

Let $(\sigma, \mathcal{H}_\sigma)$ be any irreducible, unitary representation of K . For each $\lambda \neq 0$ and $\sigma \in \widehat{K}$, we consider the representations ρ_σ^λ of \mathbb{HM} on the tensor product space $L^2(\mathbb{R}^n) \otimes \mathcal{H}_\sigma$ defined by

$$\rho_\sigma^\lambda(x, u, t, k) = \pi_\lambda(x, u, t) \mu_\lambda(k) \otimes \sigma(k)$$

where π_λ and μ_λ are the Schrödinger and metaplectic representations respectively and $(x, u, t, k) \in \mathbb{HM}$.

Proposition 4.1. *Each ρ_σ^λ is unitary and irreducible.*

Proof. It is easily seen that each ρ_σ^λ is unitary. We shall now prove that ρ_σ^λ is irreducible. Suppose $M \subset L^2(\mathbb{R}^n) \otimes \mathcal{H}_\sigma$ is invariant under all $\rho_\sigma^\lambda(x, u, t, k)$. If $M \neq \{0\}$ we will show that $M = L^2(\mathbb{R}^n) \otimes \mathcal{H}_\sigma$ proving the irreducibility of ρ_σ^λ . If M is a proper subspace of $L^2(\mathbb{R}^n) \otimes \mathcal{H}_\sigma$ invariant under $\rho_\sigma^\lambda(x, u, t, k)$ for all (x, u, t, k) , then there are nontrivial elements f and g in $L^2(\mathbb{R}^n) \otimes \mathcal{H}_\sigma$ such that $f \in M$ and g is orthogonal to $\rho_\sigma^\lambda(x, u, t, k)f$ for all (x, u, t, k) . This means that $\langle \rho_\sigma^\lambda(x, u, t, k)f, g \rangle = 0$ for all (x, u, t, k) .

Recall the functions ϕ_α^λ from Section 3.2. It is easily seen that for each $\lambda \neq 0$, $\{\phi_\alpha^\lambda : \alpha \in \mathbb{N}^n\}$ forms an orthonormal basis for $L^2(\mathbb{R}^n)$. Then, an orthonormal basis of $L^2(\mathbb{R}^n) \otimes \mathcal{H}_\sigma$ is given by $\{\phi_\alpha^\lambda \otimes e_i^\sigma : \alpha \in \mathbb{N}^n, 1 \leq i \leq d_\sigma\}$ where $\{e_i^\sigma : 1 \leq i \leq d_\sigma\}$ is an orthonormal basis of \mathcal{H}_σ and $d_\sigma = \dim \mathcal{H}_\sigma$. Now, given $f, g \in L^2(\mathbb{R}^n) \otimes \mathcal{H}_\sigma$, consider the function

$$V_g^f(x, u, t, k) = \langle \rho_\sigma^\lambda(x, u, t, k)f, g \rangle.$$

From the discussion in Section 3.3 it follows that

$$(4.1) \quad \mu_\lambda(k)\phi_\gamma^\lambda = \sum_{|\alpha|=|\gamma|} \eta_{\alpha\gamma}^\lambda(k)\phi_\alpha^\lambda$$

where $\eta_{\alpha\gamma}^\lambda$'s are the matrix coefficients of μ_λ and $k \in K \subseteq U(n)$. It is to be noted that this expansion is in terms of the scaled Hermite functions ϕ_α^λ and not in terms of the modified K -Hermite functions ψ_α^λ defined in Section 3.4. So the summation is taken over the whole of $|\alpha| = |\gamma|$ and not over a particular \mathcal{P}_{ma} . Hence, it follows that

$$V_g^f(x, u, t, k) = (2\pi)^{\frac{n}{2}} |\lambda|^{-\frac{n}{2}} e^{i\lambda t} \sum_{\alpha, \beta \in \mathbb{N}^n} \sum_{1 \leq i, j \leq d_\sigma} \sum_{|\gamma|=|\alpha|} f_{\gamma, i} \overline{g_{\beta, j}} \eta_{\alpha\gamma}^\lambda(k) \phi_{\alpha\beta}^\lambda(x, u) \phi_{ji}^\sigma(k)$$

where $f = \sum_{\gamma \in \mathbb{N}^n} \sum_{1 \leq i \leq d_\sigma} f_{\gamma, i} \phi_\gamma^\lambda \otimes e_i^\sigma$, $g = \sum_{\beta \in \mathbb{N}^n} \sum_{1 \leq j \leq d_\sigma} g_{\beta, j} \phi_\beta^\lambda \otimes e_j^\sigma$ and ϕ_{ji}^σ are the matrix coefficients of σ . Then calculating the L^2 norm of V_g^f with respect to x, u we get

$$\int_{\mathbb{R}^{2n}} |V_g^f(x, u, t, k)|^2 dx du = C \sum_{\alpha, \beta \in \mathbb{N}^n} \left| \sum_{1 \leq i, j \leq d_\sigma} \sum_{|\gamma|=|\alpha|} f_{\gamma, i} \overline{g_{\beta, j}} \eta_{\alpha\gamma}^\lambda(k) \phi_{ji}^\sigma(k) \right|^2.$$

Now, for $\pi \in \widehat{K}$, if \mathcal{H}_π is the representation space of π and $v_1, v_2, \dots, v_{d_\pi}$ is a basis of \mathcal{H}_π , then for complex numbers $c_i, 1 \leq i \leq d_\pi$ and $u \in K$, we have

$$\begin{aligned}
 & \sum_{q=1}^{d_\pi} \left| \sum_{i=1}^{d_\pi} c_i \phi_{qi}^\pi(u) \right|^2 \\
 = & \sum_{q=1}^{d_\pi} \sum_{i=1}^{d_\pi} c_i \phi_{qi}^\pi(u) \sum_{a=1}^{d_\pi} \overline{c_a \phi_{qa}^\pi(u)} \\
 (4.2) \quad = & \sum_{i,a=1}^{d_\pi} c_i \overline{c_a} \sum_{q=1}^{d_\pi} \langle \pi(u) v_i, v_q \rangle \langle v_q, \pi(u) v_a \rangle \\
 = & \sum_{i,a=1}^{d_\pi} c_i \overline{c_a} \langle \pi(u) v_i, \pi(u) v_a \rangle \\
 = & \sum_{i=1}^{d_\pi} |c_i|^2,
 \end{aligned}$$

since π is a unitary representation of K . Hence we obtain

$$\int_{\mathbb{R}^{2n}} |V_g^f(x, u, t, k)|^2 dx du = C \sum_{\gamma, \beta \in \mathbb{N}^n} \left| \sum_{1 \leq i, j \leq d_\sigma} f_{\gamma, i} \overline{g_{\beta, j}} \phi_{ji}^\sigma(k) \right|^2.$$

Integrating over K we get that

$$\begin{aligned}
 \int_K \int_{\mathbb{R}^{2n}} |V_g^f(x, u, t, k)|^2 dx du dk &= C \left(\sum_{\gamma \in \mathbb{N}^n} \sum_{1 \leq i \leq d_\sigma} |f_{\gamma, i}|^2 \right) \left(\sum_{\beta \in \mathbb{N}^n} \sum_{1 \leq j \leq d_\sigma} |g_{\beta, j}|^2 \right) \\
 &= C \|f\|^2 \|g\|^2.
 \end{aligned}$$

Under our assumption that M is nontrivial and proper, we have $V_g^f = 0$ which means that $\|f\|^2 \|g\|^2 = 0$. This is a contradiction since both f and g are nontrivial. Hence M has to be the whole of $L^2(\mathbb{R}^n) \otimes \mathcal{H}_\sigma$ and this proves that ρ_σ^λ is irreducible. \square

We now show that the representations ρ_σ^λ are enough for the Plancherel theorem. Given $f \in L^1 \cap L^2(\mathbb{H}\mathbb{M})$ consider the group Fourier transform

$$\begin{aligned}
 \widehat{f}(\lambda, \sigma) &= \int_K \int_{\mathbb{R}} \int_{\mathbb{R}^{2n}} f(x, u, t, k) \rho_\sigma^\lambda(x, u, t, k) dx du dt dk \\
 &= \int_K \int_{\mathbb{R}^{2n}} f^\lambda(x, u, k) (\pi_\lambda(x, u) \mu_\lambda(k) \otimes \sigma(k)) dx du dk.
 \end{aligned}$$

Theorem 4.2. (Plancherel) For $f \in L^1 \cap L^2(\mathbb{H}\mathbb{M})$ we have

$$\int_K \int_{\mathbb{H}^n} |f(x, u, t, k)|^2 dx du dt dk = (2\pi)^{-n} \sum_{\sigma \in \widehat{K}} d_\sigma \int_{\mathbb{R} \setminus \{0\}} \left\| \widehat{f}(\lambda, \sigma) \right\|_{HS}^2 |\lambda|^n d\lambda.$$

Proof. We calculate the Hilbert-Schmidt operator norm of $\widehat{f}(\lambda, \sigma)$ by using the basis $\{\phi_\gamma^\lambda \otimes e_i^\sigma : \gamma \in \mathbb{N}^n, 1 \leq i \leq d_\sigma\}$. We have, by (4.1),

$$\widehat{f}(\lambda, \sigma)(\phi_\gamma^\lambda \otimes e_i^\sigma) = \sum_{|\alpha|=|\gamma|} \int_K \eta_{\alpha\gamma}^\lambda(k) \int_{\mathbb{R}^{2n}} f^\lambda(x, u, k) (\pi_\lambda(x, u) \phi_\alpha^\lambda \otimes \sigma(k) e_i^\sigma) dx du dk.$$

Thus

$$\begin{aligned} & \left\langle \widehat{f}(\lambda, \sigma)(\phi_\gamma^\lambda \otimes e_i^\sigma), \phi_\beta^\lambda \otimes e_j^\sigma \right\rangle \\ &= (2\pi)^{\frac{n}{2}} |\lambda|^{-\frac{n}{2}} \sum_{|\alpha|=|\gamma|} \int_K \eta_{\alpha\gamma}^\lambda(k) \int_{\mathbb{R}^{2n}} f^\lambda(x, u, k) \phi_{\alpha\beta}^\lambda(x, u) \phi_{ji}^\sigma(k) dx du dk \end{aligned}$$

so that

$$\begin{aligned} & (2\pi)^{-n} |\lambda|^n \left\| \widehat{f}(\lambda, \sigma)(\phi_\gamma^\lambda \otimes e_i^\sigma) \right\|^2 \\ &= \sum_{\beta \in \mathbb{N}^n} \sum_{1 \leq j \leq d_\sigma} \left| \sum_{|\alpha|=|\gamma|} \int_K \eta_{\alpha\gamma}^\lambda(k) \int_{\mathbb{R}^{2n}} f^\lambda(x, u, k) \phi_{\alpha\beta}^\lambda(x, u) \phi_{ji}^\sigma(k) dx du dk \right|^2 \end{aligned}$$

and

$$\begin{aligned} & (2\pi)^{-n} |\lambda|^n \left\| \widehat{f}(\lambda, \sigma) \right\|_{HS}^2 \\ &= \sum_{\beta, \gamma \in \mathbb{N}^n} \sum_{1 \leq i, j \leq d_\sigma} \left| \sum_{|\alpha|=|\gamma|} \int_K \eta_{\alpha\gamma}^\lambda(k) \int_{\mathbb{R}^{2n}} f^\lambda(x, u, k) \phi_{\alpha\beta}^\lambda(x, u) \phi_{ji}^\sigma(k) dx du dk \right|^2. \end{aligned}$$

Using Plancherel theorem for K , we get that

$$\begin{aligned} & (2\pi)^{-n} |\lambda|^n \sum_{\sigma \in \widehat{K}} d_\sigma \left\| \widehat{f}(\lambda, \sigma) \right\|_{HS}^2 \\ &= \sum_{\beta, \gamma \in \mathbb{N}^n} \int_K \left| \sum_{|\alpha|=|\gamma|} \eta_{\alpha\gamma}^\lambda(k) \int_{\mathbb{R}^{2n}} f^\lambda(x, u, k) \phi_{\alpha\beta}^\lambda(x, u) dx du \right|^2 dk. \end{aligned}$$

Applying the same arguments as in (4.2) we obtain that the above equals

$$\sum_{\alpha, \beta \in \mathbb{N}^n} \int_K \left| \int_{\mathbb{R}^{2n}} f^\lambda(x, u, k) \phi_{\alpha\beta}^\lambda(x, u) dx du \right|^2 dk.$$

Noting that $\{\phi_{\alpha\beta}^\lambda : \alpha, \beta \in \mathbb{N}^n\}$ is an orthonormal basis for $L^2(\mathbb{R}^{2n})$ we have

$$(2\pi)^{-n} |\lambda|^n \sum_{\sigma \in \widehat{K}} d_\sigma \left\| \widehat{f}(\lambda, \sigma) \right\|_{HS}^2 = \int_K \int_{\mathbb{R}^{2n}} |f^\lambda(x, u, k)|^2 dx du dk.$$

Therefore,

$$\begin{aligned} (2\pi)^{-n} \int_{\mathbb{R}} \left(\sum_{\sigma \in \widehat{K}} d_\sigma \left\| \widehat{f}(\lambda, \sigma) \right\|_{HS}^2 \right) |\lambda|^n d\lambda &= \int_{\mathbb{R}} \int_K \int_{\mathbb{R}^{2n}} |f^\lambda(x, u, k)|^2 dx du dk d\lambda \\ &= \int_{\mathbb{R}} \int_K \int_{\mathbb{R}^{2n}} |f(x, u, t, k)|^2 dx du dk dt. \end{aligned}$$

□

Since $L^1 \cap L^2(\mathbb{H}\mathbb{M})$ is dense in $L^2(\mathbb{H}\mathbb{M})$, the Fourier transform can be uniquely extended to the whole of $L^2(\mathbb{H}\mathbb{M})$ and the above Plancherel theorem holds true for the same.

4.2. Complexified representations and Paley-Wiener type theorems.

We know that the operators $\widehat{f}(\lambda, \sigma)$ act on the basis elements $\phi_\gamma^\lambda \otimes e_i^\sigma$, which gives

$$\begin{aligned} &\widehat{f}(\lambda, \sigma)(\phi_\gamma^\lambda \otimes e_i^\sigma) \\ &= \sum_{|\alpha|=|\gamma|} \int_K \eta_{\alpha\gamma}^\lambda(k') \int_{\mathbb{R}^{2n}} f^\lambda(x', u', k') (\pi_\lambda(x', u') \phi_\alpha^\lambda \otimes \sigma(k') e_i^\sigma) dx' du' dk'. \end{aligned}$$

Now, if we consider the operator $\rho_\sigma^\lambda(x, u, t, k) \widehat{f}(\lambda, \sigma)$ acting on the basis elements we get that

$$\begin{aligned} &\rho_\sigma^\lambda(x, u, t, k) \widehat{f}(\lambda, \sigma)(\phi_\gamma^\lambda \otimes e_i^\sigma) \\ &= e^{i\lambda t} \sum_{|\alpha|=|\gamma|} \int_K \eta_{\alpha\gamma}^\lambda(k') \int_{\mathbb{R}^{2n}} f^\lambda(x', u', k') (\pi_\lambda(x, u) \mu_\lambda(k) \pi_\lambda(x', u') \phi_\alpha^\lambda \otimes \\ &\quad \sigma(k) \sigma(k') e_i^\sigma) dx' du' dk'. \end{aligned}$$

Then it follows from (3.2) that the above equals

$$\begin{aligned} &e^{i\lambda t} \sum_{|\alpha|=|\gamma|} \int_K \eta_{\alpha\gamma}^\lambda(k') \int_{\mathbb{R}^{2n}} f^\lambda(x', u', k') (\pi_\lambda(x, u) \pi_\lambda(k \cdot (x', u')) \mu_\lambda(k) \phi_\alpha^\lambda \otimes \\ &\quad \sigma(k) \sigma(k') e_i^\sigma) dx' du' dk'. \end{aligned}$$

So we obtain

$$\begin{aligned}
& \rho_\sigma^\lambda(x, u, t, k) \widehat{f}(\lambda, \sigma) (\phi_\gamma^\lambda \otimes e_i^\sigma) \\
&= e^{i\lambda t} \sum_{|\alpha'|=|\alpha|} \sum_{|\alpha|=|\gamma|} \int_K \eta_{\alpha'\alpha}^\lambda(k) \eta_{\alpha\gamma}^\lambda(k') \int_{\mathbb{R}^{2n}} f^\lambda(x', u', k') (\pi_\lambda(x, u) \pi_\lambda(k \cdot (x', u'))) \phi_{\alpha'}^\lambda \\
&\quad \otimes \sigma(k) \sigma(k') e_i^\sigma dx' du' dk' \\
&= e^{i\lambda t} \sum_{|\alpha'|=|\gamma|} \int_K \eta_{\alpha'\gamma}^\lambda(kk') \int_{\mathbb{R}^{2n}} f^\lambda(x', u', k') (\pi_\lambda(x, u) \pi_\lambda(k \cdot (x', u'))) \phi_{\alpha'}^\lambda \\
&\quad \otimes \sigma(kk') e_i^\sigma dx' du' dk'.
\end{aligned}$$

Noting that the action of $K \subseteq U(n)$ on \mathbb{R}^{2n} naturally extends to an action of G on \mathbb{C}^{2n} , this action of $\rho_\sigma^\lambda(x, u, t, k) \widehat{f}(\lambda, \sigma)$ on the basis elements $\phi_\gamma^\lambda \otimes e_i^\sigma$ can clearly be analytically continued to $\mathbb{HM}_\mathbb{C} = \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \times G$ and we get that (for suitable functions f)

$$\begin{aligned}
& \rho_\sigma^\lambda(z, w, \tau, g) \widehat{f}(\lambda, \sigma) (\phi_\gamma^\lambda \otimes e_i^\sigma) \\
&= e^{i\lambda(t+is)} \sum_{|\alpha|=|\gamma|} \int_K \eta_{\alpha\gamma}^\lambda(ke^{iH}k') \int_{\mathbb{R}^{2n}} f^\lambda(x', u', k') (\pi_\lambda(x + iy, u + iv) \\
&\quad \pi_\lambda(ke^{iH} \cdot (x', u'))) \phi_{\alpha'}^\lambda \otimes \sigma(ke^{iH}k') e_i^\sigma dx' du' dk'
\end{aligned}$$

where $(z, w, \tau, g) = (x + iy, u + iv, t + is, ke^{iH}) \in \mathbb{HM}_\mathbb{C}$. We have the following theorem:

Theorem 4.3. *Let $f \in L^2(\mathbb{HM})$. Then f extends holomorphically to $\mathbb{HM}_\mathbb{C}$ with*

$$\int_{\mathbb{H}^n} |f((z, w, \tau, g)^{-1}X)|^2 dX < \infty \quad \forall (z, w, \tau, g) \in \mathbb{HM}_\mathbb{C}$$

iff

$$\sum_{\sigma \in \widehat{K}} d_\sigma \int_{\mathbb{R}} \left\| \rho_\sigma^\lambda(z, w, \tau, g) \widehat{f}(\lambda, \sigma) \right\|_{HS}^2 |\lambda|^n d\lambda < \infty.$$

In this case we also have

$$\begin{aligned}
(4.3) \quad & \int_{\mathbb{H}^n} |f((z, w, \tau, g)^{-1}X)|^2 dX \\
&= (2\pi)^{-2n} \sum_{\sigma \in \widehat{K}} d_\sigma \int_{\mathbb{R}} \left\| \rho_\sigma^\lambda(z, w, \tau, g) \widehat{f}(\lambda, \sigma) \right\|_{HS}^2 |\lambda|^n d\lambda.
\end{aligned}$$

Before we start the proof let us set up some more notation. We know that any $f \in L^2(\mathbb{H}\mathbb{M})$ can be expanded in the K variable using the Peter Weyl theorem to obtain

$$(4.4) \quad f(x, u, t, k) = \sum_{\pi \in \widehat{K}} d_\pi \sum_{i,j=1}^{d_\pi} f_{ij}^\pi(x, u, t) \phi_{ij}^\pi(k)$$

where for each $\pi \in \widehat{K}$, d_π is the degree of π , ϕ_{ij}^π 's are the matrix coefficients of π and $f_{ij}^\pi(x, u, t) = \int_K f(x, u, t, k) \overline{\phi_{ij}^\pi(k)} dk$.

Now, for $F \in L^2(\mathbb{R}^{2n})$, consider the decomposition of the function $k \mapsto F(k \cdot (x, u))$ from K to \mathbb{C} in terms of the irreducible unitary representations of K given by

$$F(k \cdot (x, u)) = \sum_{\nu \in \widehat{K}} d_\nu \sum_{p,q=1}^{d_\nu} F_\nu^{pq}(x, u) \phi_{pq}^\nu(k)$$

where $F_\nu^{pq}(x, u) = \int_K F(k \cdot (x, u)) \overline{\phi_{pq}^\nu(k)} dk$. Putting $k = e$, the identity element of K , we obtain

$$F(x, u) = \sum_{\nu \in \widehat{K}} d_\nu \sum_{p=1}^{d_\nu} F_\nu^{pp}(x, u).$$

Then it is easy to see that for $k \in K$,

$$(4.5) \quad F_\nu^{pp}(k \cdot (x, u)) = \sum_{q=1}^{d_\nu} F_\nu^{pq}(x, u) \phi_{pq}^\nu(k).$$

From the above and the fact that $f_{ij}^\pi \in L^2(\mathbb{H}^n)$ for every $\pi \in \widehat{K}$ and $1 \leq i, j \leq d_\pi$ it follows that any $f \in L^2(\mathbb{H}\mathbb{M})$ can be written as

$$f(x, u, t, k) = \sum_{\pi \in \widehat{K}} d_\pi \sum_{\nu \in \widehat{K}} d_\nu \sum_{i,j=1}^{d_\pi} \sum_{p=1}^{d_\nu} (f_{ij}^\pi)_\nu^{pp}(x, u, t) \phi_{ij}^\pi(k).$$

In the following lemma we will prove the theorem for the functions of the form $\sum_{i,j=1}^{d_\pi} \sum_{p=1}^{d_\nu} f_{ij}^{pp}(x, u, t) \phi_{ij}^\pi(k)$. Then we shall prove the orthogonality of each part with respect to the given inner product so that we can sum up in order to prove Theorem 4.3.

Lemma 4.4. *For fixed $\pi, \nu \in \widehat{K}$, Theorem 4.3 is true for functions of the form*

$$f(x, u, t, k) = \sum_{i,j=1}^{d_\pi} \sum_{p=1}^{d_\nu} f_{ij}^{pp}(x, u, t) \phi_{ij}^\pi(k)$$

where for simplicity we write $(f_{ij}^\pi)^{pp}$ as f_{ij}^{pp} .

Proof. First we assume that $f \in L^2(\mathbb{HM})$ is holomorphic on $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \times G$ with

$$\int_{\mathbb{H}^n} |f((z, w, \tau, g)^{-1} X)^2 dX| < \infty \quad \forall (z, w, \tau, g) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \times G$$

and is of the form

$$f(x, u, t, k) = \sum_{i,j=1}^{d_\pi} \sum_{p=1}^{d_\nu} f_{ij}^{pp}(x, u, t) \phi_{ij}^\pi(k).$$

For $(x, u, t, k) \in \mathbb{HM}$, we have

$$\begin{aligned} & \rho_\sigma^\lambda(x, u, t, k) \widehat{f}(\lambda, \sigma) (\phi_\gamma^\lambda \otimes e_l^\sigma) \\ &= e^{i\lambda t} \sum_{|\alpha|=|\gamma|} \int_K \eta_{\alpha\gamma}^\lambda(kk') \int_{\mathbb{R}^{2n}} f^\lambda(x', u', k') (\pi_\lambda(x, u) \pi_\lambda(k \cdot (x', u'))) \phi_\alpha^\lambda \\ & \quad \otimes \sigma(kk') e_l^\sigma dx' du' dk'. \end{aligned}$$

Making changes of variables $(x', u') \rightarrow k^{-1} \cdot (x', u')$, $k' \rightarrow k^{-1}k'$ and using the special form of f along with (4.5) we obtain that the above equals

$$\begin{aligned} & e^{i\lambda t} \sum_{|\alpha|=|\gamma|} \int_K \eta_{\alpha\gamma}^\lambda(kk') \int_{\mathbb{R}^{2n}} f^\lambda(k^{-1}(x', u'), k') (\pi_\lambda(x, u) \pi_\lambda(x', u')) \phi_\alpha^\lambda \\ & \quad \otimes \sigma(kk') e_l^\sigma dx' du' dk' \\ &= e^{i\lambda t} \sum_{|\alpha|=|\gamma|} \sum_{i,j=1}^{d_\pi} \sum_{p,q=1}^{d_\nu} \int_K \eta_{\alpha\gamma}^\lambda(k') \int_{\mathbb{R}^{2n}} (f^\lambda)_{ij}^{pq}(x', u') (\pi_\lambda(x, u) \pi_\lambda(x', u')) \phi_\alpha^\lambda \\ & \quad \otimes \sigma(k') e_l^\sigma dx' du' \phi_{ij}^\pi(k^{-1}k') \phi_{pq}^\nu(k^{-1}) dk'. \end{aligned}$$

Then, for $(z, w, \tau, g) = (x + iy, u + iv, t + is, ke^{iH}) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \times G$, we get that

$$\begin{aligned} & \rho_\sigma^\lambda(z, w, \tau, g) \widehat{f}(\lambda, \sigma) (\phi_\gamma^\lambda \otimes e_l^\sigma) \\ &= e^{i\lambda(t+is)} \sum_{|\alpha|=|\gamma|} \sum_{i,j=1}^{d_\pi} \sum_{p,q=1}^{d_\nu} \int_K \int_{\mathbb{R}^{2n}} (f^\lambda)_{ij}^{pq}(x', u') (\pi_\lambda(x + iy, u + iv) \pi_\lambda(x', u')) \phi_\alpha^\lambda \\ & \quad \otimes \sigma(k') e_l^\sigma dx' du' \eta_{\alpha\gamma}^\lambda(k') \phi_{ij}^\pi((ke^{iH})^{-1}k') \phi_{pq}^\nu((ke^{iH})^{-1}) dk'. \end{aligned}$$

Thus, expanding the inner product $\langle \pi_\lambda(z, w) \pi_\lambda(x', u') \phi_\alpha^\lambda, \phi_\beta^\lambda \rangle$ in terms of ϕ_δ^λ and using (3.1), we have

$$\begin{aligned} & \left\langle \rho_\sigma^\lambda(z, w, \tau, g) \widehat{f}(\lambda, \sigma) (\phi_\gamma^\lambda \otimes e_l^\sigma), \phi_\beta^\lambda \otimes e_m^\sigma \right\rangle \\ &= (2\pi)^n |\lambda|^{-n} e^{i\lambda(t+is)} \sum_{\delta \in \mathbb{N}^n} \sum_{|\alpha|=|\gamma|} \sum_{i,j=1}^{d_\pi} \sum_{p,q=1}^{d_\nu} \int_K \int_{\mathbb{R}^{2n}} (f^\lambda)_{ij}^{pq}(x', u') \phi_{\alpha\delta}^\lambda(x', u') \phi_{\delta\beta}^\lambda(z, w) \\ & \quad \phi_{ml}^\sigma(k') dx' du' \eta_{\alpha\gamma}^\lambda(k') \phi_{ij}^\pi(e^{-iH} k^{-1} k') \phi_{pq}^\nu(e^{-iH} k^{-1}) dk' \end{aligned}$$

so that

$$\begin{aligned} & (2\pi)^{-2n} |\lambda|^{2n} \left\| \rho_\sigma^\lambda(z, w, \tau, g) \widehat{f}(\lambda, \sigma) (\phi_\gamma^\lambda \otimes e_l^\sigma) \right\|^2 \\ &= \sum_{\beta \in \mathbb{N}^n} \sum_{1 \leq m \leq d_\sigma} e^{-2\lambda s} \left| \sum_{p,q=1}^{d_\nu} \sum_{\delta \in \mathbb{N}^n} \sum_{|\alpha|=|\gamma|} \sum_{i,j=1}^{d_\pi} \left(\int_{\mathbb{R}^{2n}} (f^\lambda)_{ij}^{pq}(x', u') \phi_{\alpha\delta}^\lambda(x', u') dx' du' \right) \right. \\ & \quad \left. \left(\int_K \phi_{ml}^\sigma(k') \eta_{\alpha\gamma}^\lambda(k') \phi_{ij}^\pi(e^{-iH} k^{-1} k') dk' \right) \phi_{pq}^\nu(e^{-iH} k^{-1}) \phi_{\delta\beta}^\lambda(z, w) \right|^2, \end{aligned}$$

and summing over γ and l we have

$$\begin{aligned} & (2\pi)^{-2n} |\lambda|^{2n} \left\| \rho_\sigma^\lambda(z, w, \tau, g) \widehat{f}(\lambda, \sigma) \right\|_{HS}^2 \\ &= \sum_{\gamma, \beta \in \mathbb{N}^n} \sum_{1 \leq l, m \leq d_\sigma} e^{-2\lambda s} \left| \sum_{p,q=1}^{d_\nu} \sum_{\delta \in \mathbb{N}^n} \sum_{|\alpha|=|\gamma|} \sum_{i,j=1}^{d_\pi} \left(\int_{\mathbb{R}^{2n}} (f^\lambda)_{ij}^{pq}(x', u') \phi_{\alpha\delta}^\lambda(x', u') dx' du' \right) \right. \\ & \quad \left. \left(\int_K \phi_{ml}^\sigma(k') \eta_{\alpha\gamma}^\lambda(k') \phi_{ij}^\pi(e^{-iH} k^{-1} k') dk' \right) \phi_{pq}^\nu(e^{-iH} k^{-1}) \phi_{\delta\beta}^\lambda(z, w) \right|^2. \end{aligned}$$

Using Plancherel theorem for K we derive that

$$\begin{aligned} & (2\pi)^{-2n} |\lambda|^{2n} \sum_{\sigma \in \widehat{K}} d_\sigma \left\| \rho_\sigma^\lambda(z, w, \tau, g) \widehat{f}(\lambda, \sigma) \right\|_{HS}^2 \\ &= e^{-2\lambda s} \sum_{\gamma, \beta \in \mathbb{N}^n} \int_K \left| \sum_{p,q=1}^{d_\nu} \phi_{pq}^\nu(e^{-iH} k^{-1}) \sum_{\delta \in \mathbb{N}^n} \phi_{\delta\beta}^\lambda(z, w) \sum_{|\alpha|=|\gamma|} \sum_{i,j=1}^{d_\pi} \left\langle (f^\lambda)_{ij}^{pq}, \overline{\phi_{\alpha\delta}^\lambda} \right\rangle \right. \\ & \quad \left. \eta_{\alpha\gamma}^\lambda(k') \phi_{ij}^\pi(e^{-iH} k^{-1} k') \right|^2 dk'. \end{aligned}$$

Applying the same arguments as in (4.2) and change of variables $k' \rightarrow kk'$ we obtain that the above equals

$$\begin{aligned}
& e^{-2\lambda s} \sum_{\alpha, \beta \in \mathbb{N}^n} \int_K \left| \sum_{p, q=1}^{d_\nu} \phi_{pq}^\nu(e^{-iH} k^{-1}) \sum_{\delta \in \mathbb{N}^n} \phi_{\delta\beta}^\lambda(z, w) \sum_{i, j=1}^{d_\pi} \left\langle (f^\lambda)_{ij}^{pq}, \overline{\phi_{\alpha\delta}^\lambda} \right\rangle \phi_{ij}^\pi(e^{-iH} k') \right|^2 dk' \\
&= e^{-2\lambda s} \sum_{\alpha, \beta \in \mathbb{N}^n} \int_K \left| \sum_{p, q=1}^{d_\nu} \phi_{pq}^\nu(e^{-iH} k^{-1}) \sum_{\delta \in \mathbb{N}^n} \phi_{\delta\beta}^\lambda(z, w) \sum_{i, j, b=1}^{d_\pi} \left\langle (f^\lambda)_{ij}^{pq}, \overline{\phi_{\alpha\delta}^\lambda} \right\rangle \right. \\
&\quad \left. \phi_{ib}^\pi(e^{-iH}) \phi_{bj}^\pi(k') \right|^2 dk'.
\end{aligned}$$

Now, using Schur's orthogonality relations we get that

$$\begin{aligned}
& (2\pi)^{-2n} |\lambda|^{2n} \sum_{\sigma \in \widehat{K}} d_\sigma \left\| \rho_\sigma^\lambda(z, w, \tau, g) \widehat{f}(\lambda, \sigma) \right\|_{HS}^2 \\
&= \frac{e^{-2\lambda s}}{d_\pi} \sum_{\alpha, \beta \in \mathbb{N}^n} \sum_{j, b=1}^{d_\pi} \left| \sum_{p, q=1}^{d_\nu} \phi_{pq}^\nu(e^{-iH} k^{-1}) \sum_{\delta \in \mathbb{N}^n} \phi_{\delta\beta}^\lambda(z, w) \sum_{i=1}^{d_\pi} \left\langle (f^\lambda)_{ij}^{pq}, \overline{\phi_{\alpha\delta}^\lambda} \right\rangle \phi_{ib}^\pi(e^{-iH}) \right|^2.
\end{aligned}$$

Hence we have

$$\begin{aligned}
& (2\pi)^{-2n} \sum_{\sigma \in \widehat{K}} d_\sigma \int_{\mathbb{R}} \left\| \rho_\sigma^\lambda(z, w, \tau, g) \widehat{f}(\lambda, \sigma) \right\|_{HS}^2 |\lambda|^n d\lambda \\
(4.6) \quad &= \int_{\mathbb{R}} \sum_{\alpha, \beta \in \mathbb{N}^n} \sum_{j, b=1}^{d_\pi} \left| \sum_{p, q=1}^{d_\nu} \phi_{pq}^\nu(e^{-iH} k^{-1}) \sum_{\delta \in \mathbb{N}^n} \phi_{\delta\beta}^\lambda(z, w) \sum_{i=1}^{d_\pi} \left\langle (f^\lambda)_{ij}^{pq}, \overline{\phi_{\alpha\delta}^\lambda} \right\rangle \right. \\
&\quad \left. \phi_{ib}^\pi(e^{-iH}) \right|^2 \frac{e^{-2\lambda s}}{d_\pi} |\lambda|^{-n} d\lambda.
\end{aligned}$$

We have obtained an expression for one part of Lemma 4.4. Now, looking at the other part, we have

$$\begin{aligned}
& f((x, u, t, k)^{-1}(x', u', t', k')) \\
&= f\left(k^{-1}(x' - x, u' - u), t' - t - \frac{1}{2}(u \cdot x' - x \cdot u'), k^{-1}k'\right) \\
&= \sum_{i, j=1}^{d_\pi} \sum_{p, q=1}^{d_\nu} f_{ij}^{pq} \left(x' - x, u' - u, t' - t - \frac{1}{2}(u \cdot x' - x \cdot u') \right) \phi_{pq}^\nu(k^{-1}) \phi_{ij}^\pi(k^{-1}k').
\end{aligned}$$

Since f is holomorphic on $\mathbb{HM}_{\mathbb{C}}$, each f_{ij}^{pq} also have a holomorphic extension to $\mathbb{HM}_{\mathbb{C}}$. For $(z, w, \tau, g) = (x + iy, u + iv, t + is, ke^{iH}) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \times G$, we get

$$\begin{aligned} & f((z, w, \tau, g)^{-1}(x', u', t', k')) \\ &= \sum_{i,j=1}^{d_\pi} \sum_{p,q=1}^{d_\nu} f_{ij}^{pq} \left(x' - z, u' - w, t' - \tau - \frac{1}{2}(w \cdot x' - z \cdot u') \right) \phi_{pq}^\nu(e^{-iH} k^{-1}) \\ & \quad \phi_{ij}^\pi(e^{-iH} k^{-1} k'). \end{aligned}$$

Taking the L^2 -norm with respect to k' and applying change of variables $k' \rightarrow kk'$ and Schur's orthogonality relations, we obtain

$$\begin{aligned} & \int_K |f((z, w, \tau, g)^{-1}(x', u', t', k'))|^2 dk' \\ &= \int_K \left| \sum_{i,j,l=1}^{d_\pi} \sum_{p,q=1}^{d_\nu} f_{ij}^{pq} \left(x' - z, u' - w, t' - \tau - \frac{1}{2}(w \cdot x' - z \cdot u') \right) \phi_{pq}^\nu(e^{-iH} k^{-1}) \right. \\ & \quad \left. \phi_{il}^\pi(e^{-iH}) \phi_{lj}^\pi(k') \right|^2 dk' \\ &= \frac{1}{d_\pi} \sum_{j,l=1}^{d_\pi} \left| \sum_{i=1}^{d_\pi} \sum_{p,q=1}^{d_\nu} f_{ij}^{pq} \left(x' - z, u' - w, t' - \tau - \frac{1}{2}(w \cdot x' - z \cdot u') \right) \phi_{pq}^\nu(e^{-iH} k^{-1}) \right. \\ & \quad \left. \phi_{il}^\pi(e^{-iH}) \right|^2. \end{aligned}$$

Now, integrating over t' we derive that

$$\begin{aligned} & \int_{\mathbb{R}} \int_K |f((z, w, \tau, g)^{-1}(x', u', t', k'))|^2 dk' dt' \\ &= \frac{1}{d_\pi} \sum_{j,l=1}^{d_\pi} \int_{\mathbb{R}} \left| \sum_{i=1}^{d_\pi} \sum_{p,q=1}^{d_\nu} (f_{ij}^{pq})^\lambda(x' - z, u' - w) \phi_{pq}^\nu(e^{-iH} k^{-1}) \phi_{il}^\pi(e^{-iH}) \right|^2 \\ & \quad e^{2\lambda(-s - \frac{1}{2}(v \cdot x' - y \cdot u'))} d\lambda. \end{aligned}$$

So we have

$$\begin{aligned} & \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}} \int_K |f((z, w, \tau, g)^{-1}(x', u', t', k'))|^2 dk' dt' dx' du' \\ &= \frac{1}{d_\pi} \sum_{j,l=1}^{d_\pi} \int_{\mathbb{R}} e^{-2\lambda s} \int_{\mathbb{R}^{2n}} \left| \sum_{i=1}^{d_\pi} \sum_{p,q=1}^{d_\nu} (f_{ij}^{pq})^\lambda(x' - z, u' - w) \phi_{pq}^\nu(e^{-iH} k^{-1}) \phi_{il}^\pi(e^{-iH}) \right|^2 \\ & \quad e^{\lambda(u' \cdot y - v' \cdot x)} dx' du' d\lambda. \end{aligned}$$

From using (3.1) it follows that

$$\overline{\phi_{\alpha\delta}^\lambda(x, u)} = \phi_{\delta\alpha}^\lambda(-x, -u).$$

So, we can expand $(f_{ij}^{pq})^\lambda$ in terms of the orthonormal basis $\overline{\phi_{\alpha\delta}^\lambda}$ to get

$$(f_{ij}^{pq})^\lambda(x, u) = \sum_{\alpha, \delta \in \mathbb{N}^n} \left\langle (f^\lambda)_{ij}^{pq}, \overline{\phi_{\alpha\delta}^\lambda} \right\rangle \phi_{\delta\alpha}^\lambda(-x, -u)$$

and hence we have

$$(f_{ij}^{pq})^\lambda(x' - z, u' - w) = \sum_{\alpha, \delta \in \mathbb{N}^n} \left\langle (f^\lambda)_{ij}^{pq}, \overline{\phi_{\alpha\delta}^\lambda} \right\rangle \phi_{\delta\alpha}^\lambda(z - x', w - u').$$

Again, using (3.1) and the orthonormality of ϕ_β^λ we obtain that the above equals

$$|\lambda|^{-\frac{n}{2}} e^{\frac{i\lambda}{2}(z \cdot u' - w \cdot x')} \sum_{\alpha, \beta, \delta \in \mathbb{N}^n} \left\langle (f^\lambda)_{ij}^{pq}, \overline{\phi_{\alpha\delta}^\lambda} \right\rangle \phi_{\delta\beta}^\lambda(z, w) \phi_{\beta\alpha}^\lambda(-x', -u').$$

Hence we get that

$$\begin{aligned} & \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}} \int_K |f((z, w, \tau, g)^{-1}(x', u', t', k'))|^2 dk' dt' dx' du' \\ &= \frac{1}{d_\pi} \sum_{j, l=1}^{d_\pi} \int_{\mathbb{R}} |\lambda|^{-n} e^{-2\lambda s} \int_{\mathbb{R}^{2n}} \left| \sum_{i=1}^{d_\pi} \sum_{p, q=1}^{d_\nu} \sum_{\alpha, \beta, \delta \in \mathbb{N}^n} \left\langle (f^\lambda)_{ij}^{pq}, \overline{\phi_{\alpha\delta}^\lambda} \right\rangle \phi_{\delta\beta}^\lambda(z, w) \phi_{\beta\alpha}^\lambda(-x', -u') \right. \\ & \quad \left. \phi_{pq}^\nu(e^{-iH} k^{-1}) \phi_{il}^\pi(e^{-iH}) \right|^2 dx' du' d\lambda \\ &= \frac{1}{d_\pi} \sum_{j, l=1}^{d_\pi} \sum_{\alpha, \beta \in \mathbb{N}^n} \int_{\mathbb{R}} |\lambda|^{-n} e^{-2\lambda s} \left| \sum_{i=1}^{d_\pi} \sum_{p, q=1}^{d_\nu} \sum_{\delta \in \mathbb{N}^n} \left\langle (f^\lambda)_{ij}^{pq}, \overline{\phi_{\alpha\delta}^\lambda} \right\rangle \phi_{\delta\beta}^\lambda(z, w) \right. \\ & \quad \left. \phi_{pq}^\nu(e^{-iH} k^{-1}) \phi_{il}^\pi(e^{-iH}) \right|^2 d\lambda. \end{aligned}$$

From (4.6) and above we obtain the required equality.

For the converse, it is enough to prove the holomorphicity of f which in turn follows from the holomorphicity of $(f^\lambda)_{ij}^{pq}$ and the equality follows from the above argument. Assume that

$$\sum_{\sigma \in \widehat{K}} d_\sigma \int_{\mathbb{R}} \left\| \rho_\sigma^\lambda(z, w, \tau, g) \widehat{f}(\lambda, \sigma) \right\|_{HS}^2 |\lambda|^n d\lambda < \infty \quad \forall (z, w, \tau, g) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \times G.$$

From the above it is clear that for every $1 \leq l, j \leq d_\pi$ and λ almost everywhere

$$\sum_{\alpha, \beta \in \mathbb{N}^n} \left| \sum_{i=1}^{d_\pi} \sum_{p, q=1}^{d_\nu} \sum_{\delta \in \mathbb{N}^n} \left\langle (f^\lambda)_{ij}^{pq}, \overline{\phi_{\alpha\delta}^\lambda} \right\rangle \phi_{\delta\beta}^\lambda(z, w) \phi_{pq}^\nu(e^{-iH} k^{-1}) \phi_{il}^\pi(e^{-iH}) \right|^2 < \infty$$

for all $(z, w) \in \mathbb{C}^n \times \mathbb{C}^n$ and $ke^{iH} \in G$. We can put $e^{iH} = I$, the identity of the group to get

$$\sum_{\alpha, \beta \in \mathbb{N}^n} \left| \sum_{p, q=1}^{d_\nu} \sum_{\delta \in \mathbb{N}^n} \left\langle (f^\lambda)_{lj}^{pq}, \overline{\phi_{\alpha\delta}^\lambda} \right\rangle \phi_{\delta\beta}^\lambda(z, w) \phi_{pq}^\nu(k) \right|^2 < \infty$$

for all $(z, w) \in \mathbb{C}^n \times \mathbb{C}^n$ and $k \in K$. Integrating over K and using Schur's orthogonality relations we have

$$\begin{aligned} & \sum_{\alpha, \beta \in \mathbb{N}^n} \int_K \left| \sum_{p, q=1}^{d_\nu} \sum_{\delta \in \mathbb{N}^n} \left\langle (f^\lambda)_{lj}^{pq}, \overline{\phi_{\alpha\delta}^\lambda} \right\rangle \phi_{\delta\beta}^\lambda(z, w) \phi_{pq}^\nu(k) \right|^2 dk \\ &= \sum_{\alpha, \beta \in \mathbb{N}^n} \sum_{p, q=1}^{d_\nu} \left| \sum_{\delta \in \mathbb{N}^n} \left\langle (f^\lambda)_{lj}^{pq}, \overline{\phi_{\alpha\delta}^\lambda} \right\rangle \phi_{\delta\beta}^\lambda(z, w) \right|^2 < \infty. \end{aligned}$$

Hence we derive that for each $1 \leq p, q \leq d_\nu$, $1 \leq l, j \leq d_\pi$ and $(z, w) \in \mathbb{C}^n \times \mathbb{C}^n$

$$\sum_{\alpha, \beta \in \mathbb{N}^n} \left| \sum_{\delta \in \mathbb{N}^n} \left\langle (f^\lambda)_{lj}^{pq}, \overline{\phi_{\alpha\delta}^\lambda} \right\rangle \phi_{\delta\beta}^\lambda(z, w) \right|^2 < \infty.$$

Let T be the maximal torus of $K \subseteq U(n)$. After a conjugation by an element of $U(n)$ if necessary, we can consider that $T \subseteq \mathbb{T}^n$, the n -dimensional torus which is the maximal torus of $U(n)$. Now, any element $k_\theta \in T$ can be written as $e^{i\theta} = (e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n})$ where $\theta = (\theta_1, \theta_2, \dots, \theta_n)$. Notice that some of these θ_j 's may be 0 depending on T . Using the relation (3.2) and the properties of the metaplectic representation, we have

$$\phi_{\alpha\delta}^\lambda(k_\theta \cdot (x, u)) = e^{i(\delta - \alpha) \cdot \theta} \phi_{\alpha\delta}^\lambda(x, u).$$

Moreover, for each $\nu \in \widehat{K}$, $\nu|_T$ breaks up into at most d_ν irreducible components, not necessarily distinct, which we call $\nu_1, \nu_2, \dots, \nu_m \in \mathbb{Z}^n$ (abuse of notation) such that $\nu_a(e^{i\theta}) = e^{i\nu_a \cdot \theta}$ where $1 \leq a \leq m \leq d_\nu$. Choosing appropriate basis elements,

the matrix coefficients ϕ_{ab}^ν of ν satisfy $\phi_{ab}^\nu(e^{i\theta}) = \delta_{ab}e^{i\nu_a \cdot \theta}$ where δ is the Kronecker delta. So we obtain

$$\begin{aligned}
& \left\langle (f^\lambda)_{lj}^{pq}, \overline{\phi_{\alpha\delta}^\lambda} \right\rangle \\
&= \int_T \int_{\mathbb{R}^{2n}} (f^\lambda)_{lj}^{pq}(k \cdot (x, u)) \phi_{\alpha\delta}^\lambda(k \cdot (x, u)) dx du dk \\
&= \sum_{r=1}^{d_\nu} \int_T \int_{\mathbb{R}^{2n}} (f^\lambda)_{lj}^{pr}(x, u) \phi_{qr}^\nu(e^{i\theta}) e^{i(\delta-\alpha) \cdot \theta} \phi_{\alpha\delta}^\lambda(x, u) dx du dk \\
&= \left\langle (f^\lambda)_{lj}^{pq}, \overline{\phi_{\alpha-\nu_q}^\lambda} \right\rangle \delta_{\delta, \alpha-\nu_q}.
\end{aligned}$$

Hence we get that for each $1 \leq p, q \leq d_\nu$, $1 \leq l, j \leq d_\pi$ and $(z, w) \in \mathbb{C}^n \times \mathbb{C}^n$

$$\sum_{\alpha \in \mathbb{N}^n} \left| \left\langle (f^\lambda)_{lj}^{pq}, \overline{\phi_{\alpha-\nu_q}^\lambda} \right\rangle \right|^2 \left(\sum_{\beta \in \mathbb{N}^n} \left| \phi_{\alpha-\nu_q-\beta}^\lambda(z, w) \right|^2 \right) < \infty.$$

From the orthonormality properties of $\phi_{\alpha\beta}^\lambda$ it follows that

$$(4.7) \quad \sum_{\alpha \in \mathbb{N}^n} \left| \left\langle (f^\lambda)_{lj}^{pq}, \overline{\phi_{\alpha+\nu_q}^\lambda} \right\rangle \right|^2 \phi_{\alpha\alpha}^\lambda(2iy, 2iv) < \infty.$$

Now, using the above we want to prove the holomorphicity of $(f^\lambda)_{ij}^{pq}$. We note that for $(z, w) \in \mathbb{C}^{2n}$,

$$(f^\lambda)_{ij}^{pq}(z, w) = \sum_{\alpha \in \mathbb{N}^n} \left\langle (f^\lambda)_{lj}^{pq}, \overline{\phi_{\alpha+\nu_q}^\lambda} \right\rangle \phi_{\alpha-\nu_q}^\lambda(-z, -w)$$

if the sum converges. Consider a compact set $M \subseteq \mathbb{C}^{2n}$ such that $|y|^2 + |v|^2 \leq r^2$ where $(z, w) = (x + iy, u + iv)$. We know that

$$\phi_{\alpha\alpha}^\lambda(2iy, 2iv) = Ce^{\lambda(|y|^2 + |v|^2)} L_\alpha^0(-2\lambda(|y|^2 + |v|^2))$$

for any $y, v \in \mathbb{R}^n$ where $L_\alpha^0(z) = \prod_{j=1}^n L_{\alpha_j}^0(\frac{1}{2}|z_j|^2)$. Since $\phi_{\alpha\alpha}^\lambda(2iy, 2iv)$ has exponential growth and (4.7) implies holomorphicity of $(f^\lambda)_{ij}^{pq}$ as in the previous section. \square

Proof of Theorem 4.3.

To prove the theorem, it is enough to prove the orthogonality of the components

$$\begin{aligned}
f_\pi^\nu(x, u, t, k) &= \sum_{i,j=1}^{d_\pi} \sum_{p=1}^{d_\nu} f_{ij}^{pp}(x, u, t) \phi_{ij}^\pi(k). \text{ For } \pi, \nu, \pi', \nu' \in \widehat{K}, \text{ we have} \\
& (2\pi)^{-2n} |\lambda|^{2n} \sum_{\sigma \in \widehat{K}} d_\sigma \left\langle \rho_\sigma^\lambda(z, w, \tau, g) \widehat{f_\pi^\nu}(\lambda, \sigma), \rho_\sigma^\lambda(z, w, \tau, g) \widehat{f_{\pi'}^{\nu'}}(\lambda, \sigma) \right\rangle_{HS} \\
&= (2\pi)^{-2n} |\lambda|^{2n} \sum_{\sigma \in \widehat{K}} d_\sigma \sum_{\beta, \gamma \in \mathbb{N}^n} \sum_{1 \leq l, m \leq d_\sigma} \left\langle \rho_\sigma^\lambda(z, w, \tau, g) \widehat{f_\pi^\nu}(\lambda, \sigma) (\phi_\gamma^\lambda \otimes e_l^\sigma), \phi_\beta^\lambda \otimes e_m^\sigma \right\rangle \\
& \quad \overline{\left\langle \rho_\sigma^\lambda(z, w, \tau, g) \widehat{f_{\pi'}^{\nu'}}(\lambda, \sigma) (\phi_\gamma^\lambda \otimes e_l^\sigma), \phi_\beta^\lambda \otimes e_m^\sigma \right\rangle} \\
&= \sum_{\sigma \in \widehat{K}} d_\sigma \sum_{\beta, \gamma \in \mathbb{N}^n} \sum_{1 \leq l, m \leq d_\sigma} e^{-2\lambda s} \sum_{p, q=1}^{d_\nu} \phi_{pq}^\nu(e^{-iH} k^{-1}) \sum_{p', q'=1}^{d_{\nu'}} \overline{\phi_{p'q'}^{\nu'}(e^{-iH} k^{-1})} \sum_{|\alpha|=|\gamma|, |\alpha'|=|\gamma|} \\
& \quad \sum_{\delta, \delta' \in \mathbb{N}^n} \sum_{i, j=1}^{d_\pi} \sum_{i', j'=1}^{d_{\pi'}} \left\langle (f^\lambda)_{ij}^{pq}, \phi_{\alpha\delta}^\lambda \right\rangle \overline{\left\langle (f^\lambda)_{i'j'}^{p'q'}, \phi_{\alpha'\delta'}^\lambda \right\rangle} \phi_{\delta\beta}^\lambda(z, w) \overline{\phi_{\delta'\beta}^\lambda(z, w)} \\
& \quad \left(\int_K \phi_{ml}^\sigma(k') \eta_{\alpha\gamma}^\lambda(k') \phi_{ij}^\pi(e^{-iH} k^{-1} k') dk' \right) \overline{\left(\int_K \phi_{ml}^\sigma(k_1) \eta_{\alpha'\gamma}^\lambda(k_1) \phi_{i'j'}^{\pi'}(e^{-iH} k^{-1} k_1) dk_1 \right)}.
\end{aligned}$$

By Schur's orthogonality relations we get that the above equals

$$\begin{aligned}
& \sum_{\beta, \gamma \in \mathbb{N}^n} e^{-2\lambda s} \sum_{p, q=1}^{d_\nu} \phi_{pq}^\nu(e^{-iH} k^{-1}) \sum_{p', q'=1}^{d_{\nu'}} \overline{\phi_{p'q'}^{\nu'}(e^{-iH} k^{-1})} \sum_{|\alpha|=|\gamma|, |\alpha'|=|\gamma|} \sum_{\delta, \delta' \in \mathbb{N}^n} \sum_{i, j=1}^{d_\pi} \sum_{i', j'=1}^{d_{\pi'}} \\
& \left\langle (f^\lambda)_{ij}^{pq}, \phi_{\alpha\delta}^\lambda \right\rangle \overline{\left\langle (f^\lambda)_{i'j'}^{p'q'}, \phi_{\alpha'\delta'}^\lambda \right\rangle} \phi_{\delta\beta}^\lambda(z, w) \overline{\phi_{\delta'\beta}^\lambda(z, w)} \int_K \eta_{\alpha\gamma}^\lambda(k') \phi_{ij}^\pi(e^{-iH} k^{-1} k') \\
& \overline{\eta_{\alpha'\gamma}^\lambda(k') \phi_{i'j'}^{\pi'}(e^{-iH} k^{-1} k') dk'}
\end{aligned}$$

By arguments similar to (4.2) we have that the above equals

$$\begin{aligned}
& \sum_{\alpha, \beta \in \mathbb{N}^n} e^{-2\lambda s} \sum_{p, q=1}^{d_\nu} \phi_{pq}^\nu(e^{-iH} k^{-1}) \sum_{p', q'=1}^{d_{\nu'}} \overline{\phi_{p'q'}^{\nu'}(e^{-iH} k^{-1})} \sum_{\delta, \delta' \in \mathbb{N}^n} \sum_{i, j=1}^{d_\pi} \sum_{i', j'=1}^{d_{\pi'}} \left\langle (f^\lambda)_{ij}^{pq}, \phi_{\alpha\delta}^\lambda \right\rangle \\
& \overline{\left\langle (f^\lambda)_{i'j'}^{p'q'}, \phi_{\alpha'\delta'}^\lambda \right\rangle} \phi_{\delta\beta}^\lambda(z, w) \overline{\phi_{\delta'\beta}^\lambda(z, w)} \int_K \phi_{ij}^\pi(e^{-iH} k^{-1} k') \overline{\phi_{i'j'}^{\pi'}(e^{-iH} k^{-1} k') dk'}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{\alpha, \beta \in \mathbb{N}^n} e^{-2\lambda s} \sum_{p, q=1}^{d_\nu} \phi_{pq}^\nu(e^{-iH} k^{-1}) \sum_{p', q'=1}^{d_{\nu'}} \overline{\phi_{p'q'}^{\nu'}(e^{-iH} k^{-1})} \sum_{\delta, \delta' \in \mathbb{N}^n} \sum_{i, j=1}^{d_\pi} \sum_{i', j'=1}^{d_{\pi'}} \langle (f^\lambda)_{ij}^{pq}, \phi_{\alpha\delta}^\lambda \rangle \\
&\quad \overline{\langle (f^\lambda)_{i'j'}^{p'q'}, \phi_{\alpha'\delta'}^\lambda \rangle} \phi_{\delta\beta}^\lambda(z, w) \overline{\phi_{\delta'\beta}^\lambda(z, w)} \sum_{b=1}^{d_\pi} \sum_{b'=1}^{d_{\pi'}} \phi_{ib}^\pi(e^{-iH}) \overline{\phi_{i'b'}^{\pi'}(e^{-iH})} \\
&\quad \int_K \phi_{bj}^\pi(k') \overline{\phi_{b'j'}^{\pi'}(k')} dk' \\
&= 0 \text{ if } \pi \not\cong \pi'.
\end{aligned}$$

Assume $\pi \cong \pi'$. Then

$$\begin{aligned}
&(2\pi)^{-2n} |\lambda|^{2n} \sum_{\sigma \in \widehat{K}} d_\sigma \int_K \left\langle \rho_\sigma^\lambda(z, w, \tau, ke^i H) \widehat{f_\pi^\nu}(\lambda, \sigma), \rho_\sigma^\lambda(z, w, \tau, ke^i H) \widehat{f_\pi^{\nu'}}(\lambda, \sigma) \right\rangle_{HS} dk \\
&= \sum_{\alpha, \beta \in \mathbb{N}^n} e^{-2\lambda s} \sum_{p, q, r=1}^{d_\nu} \phi_{pr}^\nu(e^{-iH}) \sum_{p', q', r'=1}^{d_{\nu'}} \overline{\phi_{p'r'}^{\nu'}(e^{-iH})} \int_K \phi_{rq}^\nu(k) \overline{\phi_{r'q'}^{\nu'}(k)} dk \sum_{\delta, \delta' \in \mathbb{N}^n} \sum_{i, j, i', j'=1}^{d_\pi} \\
&\quad \langle (f^\lambda)_{ij}^{pq}, \phi_{\alpha\delta}^\lambda \rangle \overline{\langle (f^\lambda)_{i'j'}^{p'q'}, \phi_{\alpha'\delta'}^\lambda \rangle} \phi_{\delta\beta}^\lambda(z, w) \overline{\phi_{\delta'\beta}^\lambda(z, w)} \int_K \phi_{ij}^\pi(e^{-iH} k') \overline{\phi_{i'j'}^{\pi'}(e^{-iH} k')} dk' \\
&= 0 \text{ if } \nu \not\cong \nu'.
\end{aligned}$$

This proves the orthogonality of one part. On the other hand, for $\pi, \nu, \pi', \nu' \in \widehat{K}$ and $X = (x' - z, u' - w, t' - \tau - \frac{1}{2}(w \cdot x' - z \cdot u'))$, we have

$$\begin{aligned}
&\int_K f((z, w, \tau, g)^{-1}(x', u', t', k')) \overline{f((z, w, \tau, g)^{-1}(x', u', t', k'))} dk' \\
&= \sum_{i, j=1}^{d_\pi} \sum_{i', j'=1}^{d_{\pi'}} \sum_{p, q=1}^{d_\nu} \sum_{p', q'=1}^{d_{\nu'}} f_{ij}^{pq}(X) \overline{f_{i'j'}^{p'q'}(X)} \phi_{pq}^\nu(e^{-iH} k^{-1}) \overline{\phi_{p'q'}^{\nu'}(e^{-iH} k^{-1})} \\
&\quad \sum_{b=1}^\pi \sum_{b'=1}^{\pi'} \phi_{ib}^\pi(e^{-iH}) \overline{\phi_{i'b'}^{\pi'}(e^{-iH})} \int_K \phi_{bj}^\pi(k') \overline{\phi_{b'j'}^{\pi'}(k')} dk' \\
&= 0 \text{ if } \pi \not\cong \pi'.
\end{aligned}$$

Assume $\pi \cong \pi'$. Then

$$\begin{aligned}
& \int_K \int_K f((z, w, \tau, ke^{iH})^{-1}(x', u', t', k')) \overline{f((z, w, \tau, ke^{iH})^{-1}(x', u', t', k'))} dk' dk \\
&= \sum_{i,j,i',j'=1}^{d_\pi} \sum_{p,q,b=1}^{d_\nu} \sum_{p',q',b'=1}^{d_{\nu'}} f_{ij}^{pq}(X) \overline{f_{i'j'}^{p'q'}(X)} \phi_{pb}^\nu(e^{-iH}) \overline{\phi_{p'b'}^{\nu'}(e^{-iH})} \int_K \phi_{bq}^\nu(k) \overline{\phi_{b'q'}^{\nu'}(k)} dk \\
& \quad \int_K \phi_{ij}^\pi(e^{-iH}k') \overline{\phi_{i'j'}^\pi(e^{-iH}k')} dk' \\
&= 0 \text{ if } \nu \not\cong \nu'.
\end{aligned}$$

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